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A H-N MODEL FOR THE CALCULATION OF STEADY WIND- AND DENSITY-DRIVEN CIRCULATION IN THE BALTIC SEA

I. THEORETICAL BASES. STEADY WIND-DRIVEN CIRCULATION IN A HOMOGENEOUS BASIN

1. INTRODUCTION

The Baltic Sea is a basin of relatively small size (area about 400 thousand km^2, volume about 21 thousand km^3) [5, 25]. A strongly developed shore-line is characteristic of this sea, where several gulfs can be distinguished, i. e. the Gulf of Bothnia, Gulf of Finland, Gulf of Riga, Gulf of Gdańsk, which are isolated from the Baltic forming individual basins. The sea bottom topography is diversified. Shallow-water (so-called sandbanks) and deep areas (so-called deeps) occur alternately. Although the maximum sea depth is about 460 m, its mean depth is only about 55 m [5], which reflects the variability of the relief.

The hydrological regime of the Baltic, temperature, salinity and water density fields are formed under the influence of the inflow of river and North Sea waters (about 480 and 1000 km^3 yearly, respectively), the share of which is considerable as related to the volume of the sea [5, 25].

Dynamic processes resulting from interaction with the atmosphere are highly variable due to the position of the Baltic at the interface of the oceanic and continental inflows. Some predominance is exhibited by the anemobaric systems with westerly winds [27].

The characteristic features of the Baltic Sea mentioned decide as to the difficulties in the determination of its water dynamics. Despite the great number of in situ measurements of currents, the problem of Baltic water circulation still remains unsolved. The high costs of the experimental investigations necessitate the determination of currents with the help of theoretical models. Such models provide an approximate representation of the water circulation, nevertheless, the possibilities of the estimation of currents and sea level for any anemobaric situation stimulate the development of model investigations.

The problem of steady circulation in the Baltic Sea which was assumed to be a homogeneous water body, was the subject of several papers [13, 14, 20]. Non-stationary processes in the homogeneous Baltic were modelled by Svansson [26]. Steady currents in the heterogeneous Baltic Sea were considered in papers by Kowalik and Taranowska [16], Sarkisyan et al. [22], Kowalik and Staśkiewicz [15] and Simons [23].

The present paper constitutes the first part of a two-part cycle concerning the hydrodynamic-numerical (H-N) model to calculate steady wind- and density-driven circulation in the Baltic Sea. In this paper we shall discuss theoretical bases of the model and consider wind-driven currents in the homogeneous Baltic basin. Taking advantage of the H-N model linearity we shall assess the influence of the static field of atmospheric pressure and river inflows upon the structure of the wind-driven circulation in the Baltic Sea.

The density-driven currents will be considered in the second part of the cycle [11]. We shall also compare the results of numerical computations with those of the in situ measurements of currents and sea level.

2. BASIC EQUATIONS

Steady currents in basins with small and mean depths can be described by the following system of equations [3]:

$$-\Omega v = A \frac{\partial^2 u}{\partial z^2} - \frac{1}{\varrho_w} \frac{\partial p}{\partial x} \quad (1)$$

$$\Omega u = A \frac{\partial^2 v}{\partial z^2} - \frac{1}{\varrho_w} \frac{\partial p}{\partial y} \quad (2)$$

$$\frac{\partial p}{\partial z} = -\varrho_w g \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

We assume:

$$p = p_a \tag{5}$$

$$\varrho_w A \frac{\partial u}{\partial z} = \tau_x \quad ; \qquad \varrho_w A \frac{\partial v}{\partial z} = \tau_y \tag{6}$$

$$w_\xi = u_\xi \frac{\partial \xi}{\partial x} + v_\xi \frac{\partial \xi}{\partial y} \tag{7}$$

to be boundary conditions at the sea surface ($z = \xi$). Boundary conditions at the sea bottom ($z = -H$) can be written as:

$$\varrho_w A \frac{\partial u}{\partial z} = RM_x \quad ; \qquad \varrho_w A \frac{\partial v}{\partial z} = RM_y \tag{8}$$

$$w_H = - u_H \frac{\partial H}{\partial x} - v_H \frac{\partial H}{\partial y} \tag{9}$$

where:

u, v, w — components of current velocity along the x-, y- and z-axes of the coordinate system, the origin of which lies on the free sea surface, the x-axis is directed to the east, y-axis to the west and the z-axis — vertically upwards;

U_ξ, V_ξ, W_ξ, U_H, V_H, W. — components of current velocity on the sea surface and bottom,

p — hydrostatic pressure,

p_a — atmospheric pressure at the sea surface,

ξ — ordinate of the sea surface, sea level,

ϱ_w — water density,

g — gravity,

A — coefficient of the vertical turbulent exchange of momentum,

$\Omega = 2\omega \sin \varphi$ — the Coriolis parameter, where ω 7.29×10^{-5} s^{-1} is the earth's angular velocity of rotation, φ — the latitude,

τ_x, τ_y — components of wind stress at the sea surface,

H — sea depth,

$M_x = \int_{-H}^{\xi} \varrho_0 u \, dz$

$M_y = \int_{-H}^{\xi} \varrho v \, dz$ — mass transport components along the x- and y-axes, respectively,

R — the friction coefficient at the sea bottom.

It was assumed in the above equations that the coefficient of the turbulent exchange of momentum, A, is a function of (x, y), the bottom stress components (8) being proportional to the mass transport components.

The system of equations (1—4) includes four dependent variables: u, v, w, p. The number of boundary conditions (5—9) is insufficient as there is no condition concerning velocity current components at the lateral boundary. For the system of equations (1—9) the proper lateral conditions can be fitted only for the mass transport components. Therefore it is necessary to transform the basic equations and to introduce the new dependent variables: mass transport components and sea level. It leads to the proper formulation of our boundary — value problem.

To achieve this, let us integrate hydrostatic equation (3) from the depth z to the sea surface $z = \xi$. Taking into account boundary condition (5) we obtain:

$$p = p_a + g \int_{z}^{\xi} \varrho_w \, d\eta \qquad (10)$$

Water density ϱ_w can be expressed as a sum:

$$\varrho_w = \varrho_0 + \varrho \qquad (11)$$

where:

ϱ — water density anomaly,

ϱ_0 — mean water density for the whole sea.

Substituting expression (11) in (10) we obtain:

$$p = p_a + \varrho_0 g (\xi - z) + g \int_{z}^{\xi} \varrho \, d\eta \qquad (12)$$

The integral in the above equation will be written in a different form:

$$g\int_z^{\xi}\varrho d\eta = g\int_z^{0}\varrho d\eta + g\int_0^{\xi}\varrho dz \qquad (13)$$

As the density anomalies are small ($\varrho \ll \varrho_0$) we can omit the last term on the right-hand side of equation (13). Finally, equation (10) concerning pressure p can be written as:

$$p = p_a + \varrho_0 g(\xi - z) + g\int_z^{0}\varrho d\eta \qquad (14)$$

Differentiating expression (14), consecutively with respect to x and y, we obtain equations for $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ Taking into account that $\varrho \ll \varrho_0$, i.e.:

$$\frac{1}{\varrho_w} = \frac{1}{\varrho_0 + \varrho} \approx \frac{1}{\varrho_0} \qquad (15)$$

we can write the system of equations (1—12) in the following way:

$$-\Omega v = A\frac{\partial^2 u}{\partial z^2} - \frac{1}{\varrho_0}\frac{\partial p_a}{\partial x} - g\frac{\partial \xi}{\partial x} - \frac{g}{\varrho_0}\int_z^0 \frac{\partial \varrho}{\partial x} d\eta \qquad (16)$$

$$\Omega u = A\frac{\partial^2 u}{\partial z^2} - \frac{1}{\varrho_0}\frac{\partial p_a}{\partial y} - g\frac{\partial \xi}{\partial y} - \frac{g}{\varrho_0}\int_z^0 \frac{\partial \varrho}{\partial y} d\eta \qquad (17)$$

Assuming that the atmospheric pressure p_a, sea level ξ and water density ϱ are known, equations (16) and (17) resolve into an equation for complex velocity $D = u + iv$:

$$\frac{\partial^2 D}{\partial z^2} - p_1^2 D = G \qquad (18)$$

where:

$$G = \frac{1}{A\varrho_0}\left(\frac{\partial p_a}{\partial x} + i\frac{\partial p_a}{\partial y}\right) + \frac{g}{A}\left(\frac{\partial \xi}{\partial x} + i\frac{\partial \xi}{\partial y}\right) + \frac{g}{A\varrho_0}\int_z^0\left(\frac{\partial \varrho}{\partial x} + i\frac{\partial \varrho}{\partial y}\right)d\eta \qquad (19)$$

$$p_1^2 = \frac{\Omega}{A} i \quad ; \quad i = \sqrt{-1}$$

Taking into account that $\varrho \ll \varrho_0$ we obtain boundary conditions (6) and (8) for the complex velocity *D:*

$$\varrho_0 A \frac{\partial D}{\partial z} = \tau \quad \text{for} \quad z = \xi \approx 0 \tag{20}$$

$$\varrho_0 A \frac{\partial D}{\partial z} = RM \quad \text{for} \quad z = -H \tag{21}$$

where:

$$\tau = \tau_x + i\tau_y \quad , \quad M = M_x + iM_y \tag{22}$$

Integrating equation (18) with conditions (20) and (21) we obtain the analytical expression for the calculation of velocity:

$$D = \frac{\tau}{\varrho_0 p_1 A} \frac{\mathrm{ch}\, p_1 (H+z)}{\mathrm{sh}\, p_1 H} + \Big[- \frac{RM}{\varrho_0 p_1 A} - B_1(x,y,-H) \exp(-p_1 H) -$$

$$- B_1(x,y,-H) \exp(p_1 H) \Big] \frac{\mathrm{ch}\, p_1 z}{\mathrm{sh}\, p_1 H} + B_1(x,y,z) \exp(p_1 z) +$$

$$+ B_2(x,y,z) \exp(-p_1 z) + G_1(p_a, \xi) \tag{23}$$

where:

$$G_1(p_a, \xi) = \frac{1}{\varrho_0 \Omega} i \left(\frac{\partial p_a}{\partial x} + i \frac{\partial p_a}{\partial y} \right) + \frac{g}{\Omega} i \left(\frac{\partial \xi}{\partial x} + i \frac{\partial \xi}{\partial y} \right) \tag{24}$$

$$B_1(x,y,z) = \frac{1}{2p_1} \int_0^z G_2(x,y,\eta) \exp(-p_1 \eta)\, d\eta \tag{25}$$

$$B_2(x,y,z) = -\frac{1}{2p_1} \int_0^z G_2(x,y,\eta) \exp(p_1 \eta)\, d\eta \tag{26}$$

$$G_2(x,y,z) = \frac{g}{\varrho_0 A} \int_z^0 \left(\frac{\partial \varrho}{\partial x} + i \frac{\partial \varrho}{\partial y} \right) d\eta \tag{27}$$

The problem of the assessment of circulations would have been completed if the values of the sea level and mass transport had been known. Now we shall try to find a method by which to calculate these quantities.

3. MASS TRANSPORT AND SEA LEVEL EQUATIONS

To estimate the sea level and mass transport components one can transform equations (16) and (17) in such a way that by eliminating the sea level we obtain one equation for the stream function Ψ (6, 7, 13, 14). In our model we will apply a different method using the relation between steady and non-stationary processes, created by the generating forces which are constant in time (wind, water density field). Equations (16) and (17) will be written in a non-stationary form:

$$\frac{\partial u}{\partial t} - \Omega v = A\frac{\partial^2 u}{\partial z^2} - g\frac{\partial \xi}{\partial x} - \frac{1}{\varrho_0}\frac{\partial p_a}{\partial x} - \frac{g}{\varrho_0}\int_z^0 \frac{\partial \varrho}{\partial x}\, d\eta \tag{28}$$

$$\frac{\partial v}{\partial t} + \Omega u = A\frac{\partial^2 v}{\partial z^2} - g\frac{\partial \xi}{\partial y} - \frac{1}{\varrho_0}\frac{\partial p_a}{\partial y} - \frac{g}{\varrho_0}\int_z^0 \frac{\partial \varrho}{\partial y}\, d\eta \tag{29}$$

Continuity equation (4) and boundary conditions (6, 8, 9) will be left unchanged. Boundary condition (7) for vertical velocity at the sea surface will be modified:

$$w_\xi = \frac{\partial \xi}{\partial t} + u_\xi\frac{\partial \xi}{\partial x} + v_\xi\frac{\partial \xi}{\partial y} \qquad \text{for} \quad z = \xi = 0 \tag{30}$$

As the required quantities u, v, w, ξ are now also functions of time, thus apart from boundary conditions (6, 8, 9) and (30), the initial conditions should be formulated as for example:

$$u = v = w = \xi = 0 \tag{31}$$

Integrating equations (28, 29, 4) from the sea bottom ($z = -H$) to the surface $z = \xi \approx 0$, taking into account boundary conditions (6, 8, 9, 30) and assuming that $\xi \ll H$ we obtain a system of equations for mass transport and the sea level *:

$$\frac{\partial M_x}{\partial t} - \Omega M_y = \tau_x - RM_x - \varrho_0 gH\frac{\partial \xi}{\partial x} - H\frac{\partial p_a}{\partial x} \quad S_x \tag{32}$$

$$\frac{\partial M_y}{\partial t} + \Omega M_x = \tau_y - RM_y - \varrho_0 gH\frac{\partial \xi}{\partial y} - H\frac{\partial p_o}{\partial y} \cdot S_y \tag{33}$$

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* On deriving equations (28, 29), the expressions $\varrho_0 u\frac{\partial \xi}{\partial t}$ and $\varrho_0 v\frac{\partial \xi}{\partial t}$, the, order of which is less than that of the remaining terms have been omitted.

$$\frac{\partial \xi}{\partial t} = -\frac{1}{\varrho_0}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y}\right) \tag{34}$$

where:

$$S_x = g \int_{-H}^{0}\int_{z}^{0} \frac{\partial \varrho}{\partial x}\, d\eta\, dz \tag{35}$$

$$S_y = g \int_{-H}^{0}\int_{z}^{0} \frac{\partial \varrho}{\partial y}\, d\eta\, dz \tag{36}$$

The initial conditions for the system of equations (32—34) are assumed as follows:

$$M_x = M_y = \xi = 0 \qquad \text{for} \quad t = 0 \tag{37}$$

We will formulate boundary conditions for the mass transport component normal to the contour L of the basin shore line:

$$M_n\Big|_L = \begin{cases} 0 & \text{- at the solid boundary} \\ \Phi(L) & \text{- at the liquid boundary} \end{cases} \tag{38}$$

where:

$\Phi(L)$ — water budget at the liquid boundary.

Setting the values of forces generating tangent wind stress $-\tau_x$, τ_y, water density ϱ and values of physical parameters R, A, Ω, g of the model, the system of equations (32—34) can be integrated in time. Generating forces are assumed constant in time, therefore after a certain period a steady state will be attained which is the result of the energy dissipation due to friction of liquid against the bottom and the shores of the basin. When mass transports and sea level are known from equation (23), horizontal components of the velocity of current can be calculated and from continuity equation (4) we can estimate vertical component w of current velocity.

The system of equations (32—34) for mass transports and sea level is one of differential equations including three dependent quantities, i.e. M_x, M_y and ξ. Boundary condition (38) determines the values at the sea shore form mass transport components only. Formulation of a boundary condition, which is physically proper, for the sea level, is complicated. This problem can be omitted if a special differential scheme is employed for numerical solution of the equation system (32—34, 38). However, the question arises as to whether the number of boundary conditions is sufficient for the investigated equation system to have a unique solution. We shall now deal with this problem.

4. UNIQUENESS OF THE SOLUTION OF THE EQUATION SYSTEM (32—34) FOR MASS TRANSPORTS AND SEA LEVEL

Considering the boundary problem for differential equations, we add boundary and initial conditions in order to determine a unique solution. In this connection questions arise concerning two basic features of the boundary problems:

— the existence of the solution — i.e. if a solution exists for given boundary and initial conditions, and if there are such conditions, among the ones given, which exclude each other,

— uniqueness of the solution — i.e. assuming that the solution exists one should prove that the number of conditions is sufficient for the boundary value problem to have one and only one solution.

Demonstration of the existence of the solution is strictly related to the method of solving the equations and is usually rather complicated. For the system of equations (32—34) for mass transports and sea level, the existence and uniqueness of the solution upon the application of the generalized equations theory (18) were proved. Uniqueness of the solution of equations (32—34) can easily be proved. In order to attain this let us consider circulations in a rectangular basin *ABCD* with a surface area *S* and a shore line contour *L* (Fig. 1). We shall formulate our problem in the form of a theorem.

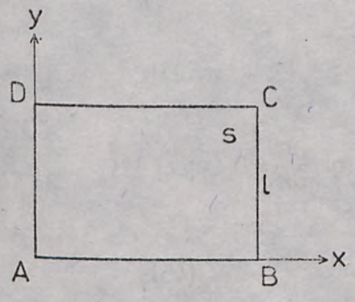

Fig. 1. Model sea basin.
Rys. 1. Modelowy akwen morski

Theorem:

If functions M_x, M_y, ξ are well-defined and continuous, together with their derivatives, in relation to space and time in area *S* within the boundary *L*, for time $t \geqslant 0$, and if the coefficients in equations (32—34) are continuous within area *S* including boundary *L*, then the equation system (32—34) with the initial (37) and boundary (38) conditions has a unique solution.

Proof:

Assume that the theorem is false and the equation system studied with given conditions has two different solutions which are symbolically denoted in the following way:

$$R_1 = (M_x^1, M_y^1, \xi^1) \qquad (39)$$

$$R_2 = (M_x^2, M_v^2, \xi^2) \qquad (40)$$

The equation system (32—34) is linear so the superposition of solutions (39) and (40) will also be its solution (4). Let us take into account a solution being the result of the difference between (39) and (40). It can be written symbolically as:

$$R^0 = (M_x^1 - M_x^2, M_y^1 - M_y^2, \xi^1 - \xi^2) \qquad (41)$$

or introducing a new notation M_x^0, M_y^0, ξ^0:

$$R^0 = (M_x^0, M_y^0, \xi^0) \qquad (42)$$

Substituting solutions R_1 and R_2 into equations (32—34) and subtracting the sides of the equations we obtain a homogeneous system of equations with which to solve R^0:

$$\frac{\partial M_x^0}{\partial t} - \Omega M_y^0 + RM_x^0 + \varrho_0 gH \frac{\partial \xi^U}{\partial x} = 0 \qquad (43)$$

$$\frac{\partial M_y^U}{\partial t} + \Omega M_x^0 + RM_y^0 + \varrho_0 gH \frac{\partial \xi^0}{\partial x} = 0 \qquad (44)$$

$$\frac{\partial \xi^U}{\partial t} = -\frac{1}{\varrho}\left(\frac{\partial M_x^0}{\partial x} + \frac{\partial M_y^0}{\partial y}\right) \qquad (45)$$

Carrying out analogical considerations for boundary conditions (37) and (38), we obtain boundary and initial conditions for the system (43—45):

$$M_x^0 = M_y^0 = \xi^0 = 0 \qquad \text{for} \quad t = 0 \qquad (46)$$

$$M_n^0\Big|_L = 0 \qquad (47)$$

We shall prove that the system of equations (43—45) with boundary conditions (46) and (47) has trivial solutions equal to zero. To do this, let us multiply equation (43) by $\frac{M^0}{\varrho^o H}$, equation (44) by $\frac{M^0}{\varrho^o H}$ and equation (45) $\varrho_o g\ \xi^0$. Adding up the sides of the equations, after simple transformations we obtain:

$$-\frac{\partial}{\partial t}\left[\frac{1}{2\varrho_0 H}(M_x^{02}+M_y^{02})+\frac{1}{2}\varrho_0 g\xi^{0}\right]-\frac{R}{\varrho_0 H}(M_x^{02}+M_y^{02})=$$

$$=g\left[\frac{\partial}{\partial x}(M_x^0\xi^0)+\frac{\partial}{\partial y}(M_y^0\xi^0)\right] \tag{48}$$

We now integrate the equation obtained in area S:

$$-\iint\limits_S\left\{\frac{\partial}{\partial t}\left[\frac{1}{2\varrho_0 H}(M_x^{02}+M_y^{02})+\frac{1}{2}\varrho_0 g\xi^{02}\right]+\frac{R}{\varrho_0 H}(M_x^{02}+M_y^{02})\right\}dx\,dy=$$

$$=\iint\limits_S g\left[\frac{\partial}{\partial x}(M_x^0\xi^0)+\frac{\partial}{\partial y}(M_y^0\xi^0)\right]dx\,dy. \tag{49}$$

Applying the Green formula to the right-hand side we obtain:

$$-\iint\limits_S\left\{\frac{\partial}{\partial t}\left[\frac{1}{2\varrho_0 H}(M_x^{02}+M_y^{02})+\frac{1}{2}\varrho_0 g\xi^{02}\right]+\frac{R}{\varrho_0 H}(M_x^{02}+M_y^{02})\right\}dx\,dy=$$

$$=g\int\limits_L(-M_y^0\xi^0dx+M_x^0\xi^0dy) \tag{50}$$

Integrating along contour L of the border line of area S and taking into account boundary condition (47) we have:

$$g\int\limits_L(-M_y^0\xi^0dx+M_x^0\xi^0dy)=0 \tag{51}$$

Integrating equation (50) in relation to time, from $t=0$ to $t=T$ and taking into account initial condition (46) and equality (51) we obtain:

$$\iint\limits_S\left\{\left[\frac{1}{2\varrho_0 H}(M_x^{02}+M_y^{02})+\frac{1}{2}\varrho_0 g\xi^{02}\right]+\int\limits_0^T\frac{R}{\varrho_0 H}(M_x^{02}+M_y^{02})dt\right\}dx\,dy=0 \tag{52}$$

Hence:

$$\frac{1}{2\varrho_0 H}(M_x^{02}+M_y^{02})+\frac{1}{2}\varrho_0 g\xi^{02}+\int\limits_0^T\frac{R}{\varrho_0 H}(M_x^{02}+M_y^{02})\,dt=0 \tag{53}$$

All components on the right-hand side are positive. Equality (53) is true when the following relations are fulfilled:

$$M_x^0 \equiv 0\,;\ M_y^0 \equiv 0\,;\ \xi^0 \equiv 0 \quad (54)$$

We have proved that our assumption of the existence of two different solutions is false. Hence, the system of equations (32—34) with the boundary conditions (37) and (38) has a unique solution.

5. ANALYSIS OF EQUATIONS OF MASS TRANSPORT AND SEA LEVEL

Let us consider a stationary form of the equation system (32—34) for mass transport and sea level:

$$RM_x - \Omega M_y = \tau_x - \varrho_0 gH\frac{\partial \xi}{\partial x} - H\frac{\partial p_a}{\partial x} - S_x \quad (55)$$

$$RM_y + \Omega M_x = \tau_x - \varrho_0 gH\frac{\partial \xi}{\partial y} - H\frac{\partial p_a}{\partial y} - S_y \quad (56)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0 \quad (57)$$

Let us define the mass transport components as those of a flat, two-dimensional vector $\vec{M}$ of mass transport. Regarding the field of the mass transport vectors, we shall employ the method of the field theory, estimating its rotation and divergence. In respect of a flat vector field, only the component of field rotation along the z-axis will exist. Therefore the rotation vector should be treated as a pseudovector. Equation (57) immediately gives the expression for the divergence of the mass transport vector field:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = \operatorname{div}\vec{M} = 0 \quad (58)$$

Hence, it can be concluded that in the field of $\vec{M}$ vectors there are neither positive nor negative sources of liquid. To calculate the z component of the rotation of the transport field we assume that the Coriolis parameter depends only on latitude [3]:

$$\Omega = \Omega^0 + \beta y \qquad (59)$$

where:

Ω^0 — a constant quantity,

$\beta = \frac{d\Omega}{dy} = \text{const.}$

We determine the mass transport components M_x, M_y from equations (55) and (56):

$$M_x = \frac{1}{R^2 + \Omega^2}\Big[R\tau_x + \Omega\tau_y - \varrho_0 gH\left(R\frac{\partial\xi}{\partial x} + \Omega\frac{\partial\xi}{\partial y}\right) - H\left(R\frac{\partial p_a}{\partial x} + \Omega\frac{\partial p_a}{\partial y}\right) - RS_x - \Omega S_y\Big] \qquad (60)$$

$$M_y = \frac{1}{R^2 + \Omega^2}\Big[-\Omega\tau_x + R\tau_y - \varrho_0 gH\left(-\Omega\frac{\partial\xi}{\partial x} + R\frac{\partial\xi}{\partial y}\right) - H\left(-\Omega\frac{\partial p_a}{\partial x} + R\frac{\partial p_a}{\partial y}\right) + \Omega S_x - RS_y\Big] \qquad (61)$$

The z-component of the rotation of the mass transport vector field can be calculated from the formula:

$$\mathrm{rot}_z\vec{M} = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y} \qquad (62)$$

Differentiating expression (60) relative to y, and (61) relative to x and substituting in formula (62) we obtain:

$$\mathrm{rot}_z\vec{M} = \frac{1}{R^2 + \Omega^2}\Big\{\underbrace{\beta(2\Omega M_x - \tau_y + \varrho_0 gH\frac{\partial\xi}{\partial y} + H\frac{\partial p_a}{\partial y} + S_y)}_{\mathrm{I}} +$$

$$+ \underbrace{\varrho_0 gH\Omega\Delta\xi}_{\mathrm{II}} + \underbrace{H\Omega\Delta p_a}_{\mathrm{III}} - \underbrace{R(\mathrm{rot}_z\vec{S} - \mathrm{rot}_z\vec{\tau})}_{\mathrm{IV}} - \underbrace{\Omega(\mathrm{div}\,\vec{\tau} - \mathrm{div}\,\vec{S})}_{\mathrm{V}} +$$

$$+ \underbrace{\Omega\nabla p_a\nabla H}_{\mathrm{VI}} + \underbrace{RJ(p_a, H)}_{\mathrm{VII}} + \underbrace{\varrho_0 g\Omega\nabla\xi\nabla H}_{\mathrm{VIII}} + \underbrace{\varrho_0 gRJ(\xi, H)}_{\mathrm{IX}} +$$

$$+\underbrace{\varrho_0 gH\,J(\xi,R)}_{X}+\underbrace{H\,J(p_a,R)}_{XI}+\underbrace{\frac{\partial R}{\partial x}(\tau_y-S_y+2RM_y)}_{XII}-$$

$$-\underbrace{\frac{\partial R}{\partial y}(\tau_x-S_x-2RM_x)}_{XIII} \tag{63}$$

where:

$\vec{S}$ — vector with the components S_x, S_y,

$\vec{\tau}$ — vector with the components τ_x, τ_y

$$J(u,v)=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}, \qquad \nabla=\vec{i}\frac{\partial}{\partial x}+\vec{k}\frac{\partial}{\partial y}$$

where $\vec{i}$, $\vec{k}$ — versors along the x- and y-axis, respectively,

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

Let us consider the individual terms in expression (63). Terms denominated by I describe the share of the so-called β-effect in the rotation of the field of mass transports. Consecutive terms, II and III, relate $rot_z\vec{M}$ to the sea level and the atmospheric pressure fields. It is known from mathematical analysis that the second derivative describes the smoothness of the function. In our considerations it concerns the smoothness of the ξ and p_a fields. The relation of the rotation of the mass transport field with the divergence and rotation of the field of generating forces, i. e. the tangent wind stress and the stress due to the heterogeneity of sea water density, is expressed by terms IV and V Subseqent terms rélate $rot_z\vec{M}$ to the gradients of atmospheric pressure, sea level and sea depths (terms VI and VIII) and to the derivatives of these quantities (terms VII and IX). The remaining terms describe the share of the variability of the friction coefficient at the bottom and its relations with the sea level and atmospheric pressure, in the rotation of the field of mass transports.

A decisive role in the formation of the field of mass transport is played the generating forces (vectors $\vec{\tau}$ and $\vec{S}$) and the sea bottom topography Therefore relation (63) can be used in a qualitative assessment of the correctness of the results of the mass transport model calculations.

6. PHYSICAL PARAMETERS AND GENERATING FORCES

As the aim of the present paper is to investigate wind-driven circulations in the homogeneous Baltic Sea, we shall omit the effects due to the heterogeneity of sea water, which will be discussed in the second part of this paper [11]. In the case of a homogeneous basin the system of equations (32—34) for mass transport and sea level can be written as:

$$\frac{\partial M_x}{\partial t} - \Omega M_y = \tau_x - RM_x - \varrho_0 gH \frac{\partial \xi}{\partial x} - H \frac{\partial p_a}{\partial x} \tag{64}$$

$$\frac{\partial M_x}{\partial t} + \Omega M_x = \tau_y - RM_y - \varrho_0 gH \frac{\partial \xi}{\partial y} - H \frac{\partial p_a}{\partial y} \tag{65}$$

$$\frac{\partial \xi}{\partial t} = -\frac{1}{\varrho_0}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y}\right) \tag{66}$$

Expression (23) used to estimate the velocity of current can be reduced to give the following expression:

$$D = \frac{\tau}{\varrho_0 p_1 A} \frac{\operatorname{ch} p_1(H+z)}{\operatorname{sh} p_1 H} - \frac{RM}{\varrho_0 p_1 A} \frac{\operatorname{ch} p_1 z}{\operatorname{sh} p_1 H} + \frac{g}{\Omega} i\left(\frac{\partial \xi}{\partial x} + i \frac{\partial \xi}{\partial y}\right)$$
$$+ \frac{1}{\varrho_0 \Omega} i\left(\frac{\partial p_a}{\partial x} + i \frac{\partial p_a}{\partial y}\right) \tag{67}$$

The system of equations (64—66) for mass transport and sea level can be solved by numerical methods only, e. g. applying the H-N differential scheme [9, 10]. Distribution of the nodes of the H-N numerical grid inscribed into the area of the Baltic Sea, with a spatial step equal to 5 nautical miles, is presented on the map in Fig. 2.

The application of the above H-N model to investigate currents in the Baltic Sea requires explanation. The model is based on hydrostatic equation (3) describing the vertical motion of liquid. Currents with high acceleration and vertical velocities can therefore be excluded from our considerations. Characteristic of the Baltic is its highly diversified bottom topography so that approximating the depth field by numerical grid, we obtain rapid fluctuations of depth on proceeding between neighbouring nodes of the grid (Fig. 3).

This will generate strong vertical circulations which are not described properly by the H-N model and may cause instability in numerical solution of the mass transport and sea level equations (32—34) or (64—66), thus causing divergence of the solution [28]. To avoid this, one should smooth out the depth field, e. g. by the following formula:

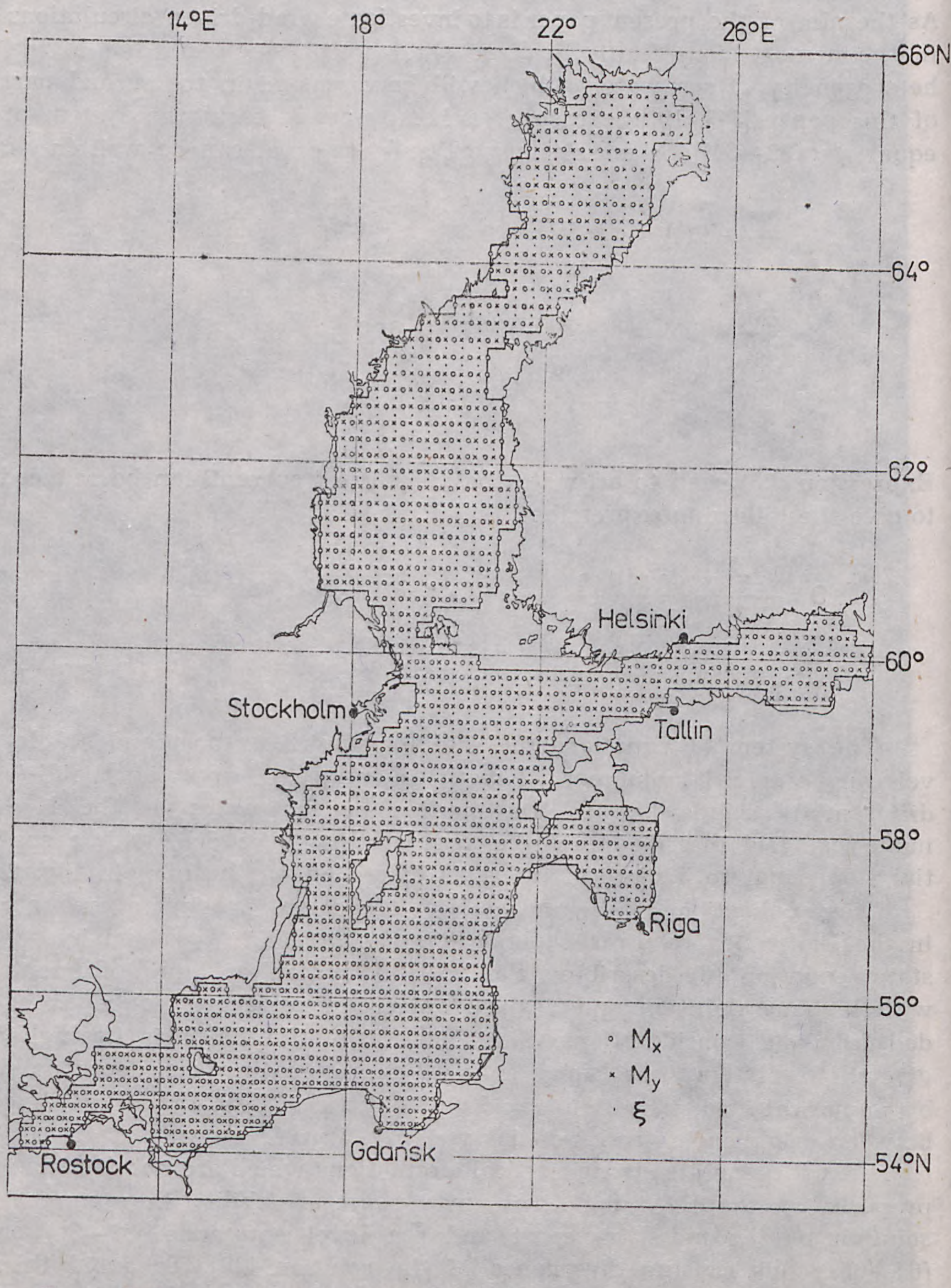

Fig. 2. H-N numerical grid for the Baltic Sea.
Rys. 2. Siatka numeryczna H-N dla Morza Bałtyckiego.

$$H^w_{m,n} = 0.125\,(H_{m+1,n} + H_{m-1,n} + H_{m,n+1} + H_{m,n-1} + 4H_{m,n}) \qquad (68)$$

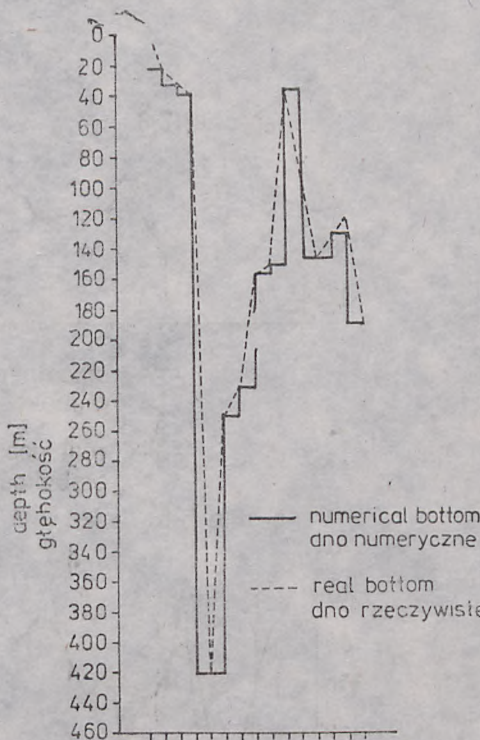

Fig. 3. Example of an approximation of the Baltic bottom topography by means of the H-N numerical grid.
Rys. 3. Przykład aproksymacji dna Bałtyku za pomocą siatki numerycznej H-N.

where:

m, n — indexes of the numerical grid,

$H_{m,n}$, $H^w_{m,n}$ — true and smoothed depth in the node (m, n).

The effect of the smoothing out the bottom by expression (68) is shown in Fig. 4 for several profiles, the distribution of which in the Baltic area is shown on a schematic map in Fig. 5. Fig. 6 shows the smoothed depth field of the Baltic Sea.

In our considerations, the coefficient of vertical turbulent exchange of momentum, A, is assumed to be a function of x, y and is calculated using the Felzenbaum theory [6]:

$$A = 0.54 \times 10^{-4} W_a H \qquad \text{if} \quad H \leq H_k \qquad (69)$$

$$A = 4.7 \times 10^{-8} \frac{W_a^2}{\Omega} \qquad \text{if} \quad H > H_k \qquad (70)$$

where:

W_a — modulus of wind velocity,

$H_k = 8.7 \times 10^{-4} \frac{W_a}{\Omega}$ is the depth separating shallow and deep the sea areas.

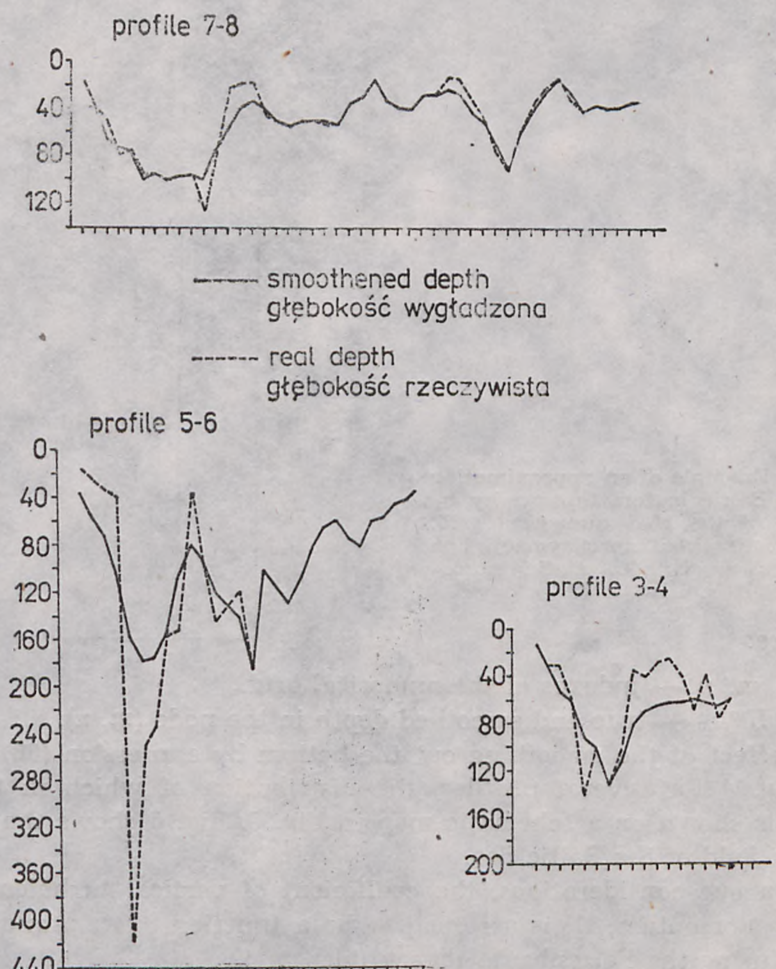

Fig. 4. Effect of smoothing of the depth field according to expression (68) for selected depth profiles.
Rys. 4. Efekt wygładzania pola głębokości według wzoru (68) dla wybranych profili.

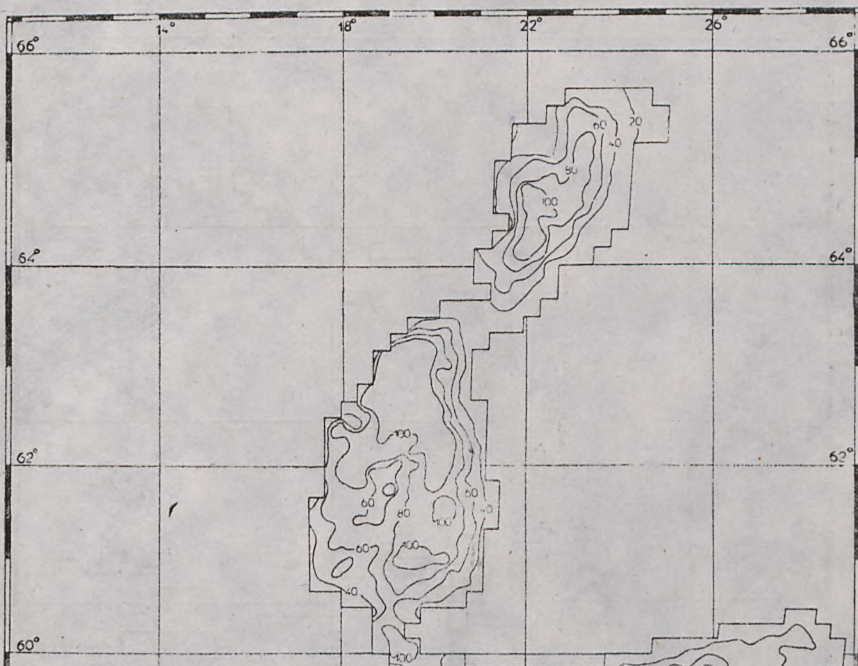

Fig. 5. Distribution of depth profiles in the Baltic.
Rys. 5. Rozmieszczenie profili głębokościowych w obszarze Bałtyku.

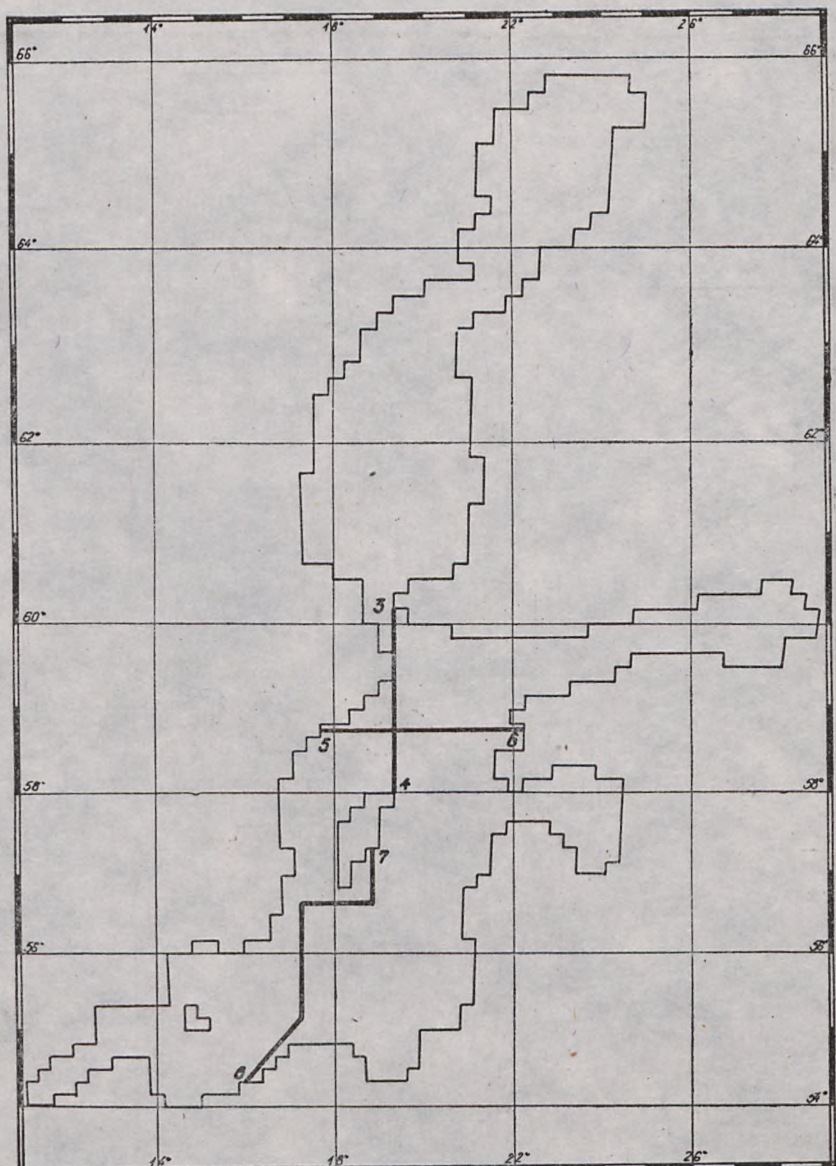

Fig. 6. Depth field of the Baltic Sea prepared for the calculations using the H-N model. Numbers on the isobaths indicate the sea depth in metres.
Rys. 6. Pole głębokości Morza Bałtyckiego przygotowane do obliczeń za pomocą modelu H-N. Liczby przy izobatach oznaczają głębokości morza w metrach.

The bottom friction coefficient, R, can be estimated using the analytical relation [13, 28]:

$$R = \frac{\pi A}{4H^2} \qquad (71)$$

The determination of wind stress components τ_x, τ_y, which constitute the generating forces in equations (64) and (65), needs some explanation. In model investigations of wind-driven currents, formulae based on the Akelbroom model [3, 21] were widely applied:

$$\tau_x = -\sqrt{\frac{A_a}{2\varrho_a\Omega}}\left(\frac{\partial p_a}{\partial x} + \frac{\partial p_a}{\partial y}\right) \qquad (72)$$

$$\tau_y = -\sqrt{\frac{A_a}{2\varrho_a\Omega}}\left(\frac{\partial p_a}{\partial x} - \frac{\partial p_a}{\partial y}\right) \qquad (73)$$

where:

ϱ_a — air density,

A_a — coefficient of the vertical turbulent exchange of momentum for the atmosphere.

In our opinion, the application of these formulae presents certain difficulties. The problem consists in the determination of the coefficient A_a of turbulent exchange of momentum, as well as wind stress components τ_x, τ_y in the case of winds of constant velocities (p_a = const). The latter problem concerns, in particular, small sea basins where the constant velocity approximation is acceptable. The next difficulty arises from the fact that expressions (72) and (73) give the deviation angle β for the wind stress vector from the isobar, equal to 45°. Investigations of dynamic processes in the atmospheric boundary layer above the sea indicate that angle is smaller and is of the order of 15 — 20° [30].

These problems can, to some extent, be solved by employing a different way of estimating τ_x, τ_y, i. e. using the analytical dependence of tangent stress upon the wind velocity [6, 29]:

$$\tau_x = \gamma W_a W_x \qquad (74)$$

$$\tau_y = \gamma W_a W_y \qquad (75)$$

where:

W_x, W_y — wind velocity components along the x- and y-axis, respectively,

$\gamma = 3.25 \times 10^{-6}$.

To estimate the wind velocity above the sea we use formulae for the calculation of geostrophic wind velocity [8, 24]:

$$W_a = B \frac{dp_a}{dn} \quad (76)$$

where:

$\frac{dp_a}{d_n}$ = horizontal gradient of atmospheric pressure (in m/100 km).

We assume coefficient B to depend only upon the difference in temperature t_w and t_a of water and air, respectively, at the sea-atmosphere interface, the data being taken from paper [8].

Values of coefficient B

$\Delta = t_a - t_w$	$\Delta < -2.0°$	$-2.0° \leqslant \Delta \leqslant 0.0°$	$0.1° \leqslant \Delta \leqslant 0.5°$	$\Delta > 0.5°$
B	5.17	4.97	4.66	3.95

In calculations of a heterogeneous wind field, the mean atmospheric pressure field obtained over a period of many years for August (Fig.7) and elaborated by the Maritime Branch of the Institute of Meteorology and Water Management in Gdynia, were used. The field of water temperature on the sea surface was taken from the atlas by Lenz [17], and the field of water temperature above the sea was obtained from Prof. F. Defant from the Kiel University.

The geostrophic wind velocity was calculated by means of expression (76) and assuming the angle of deviation of the wind velocity vector from the isobar to be $\beta = 15°$, the wind field over the Baltic for August was obtained (Fig. 8).

Low wind velocities are characteristic of the wind field calculated, the maximum velocities not exceeding 5 m s^{-1}. West and south-west winds prevail. South winds predominate only in the northern part of the Gulf of Bothnia.

7. BAROGRADIENT EFFECT

Analyzing equations (64) and (65) we can see that besides the tangent wind stress forces, terms involving atmospheric pressure are also factors which generate the circulation:

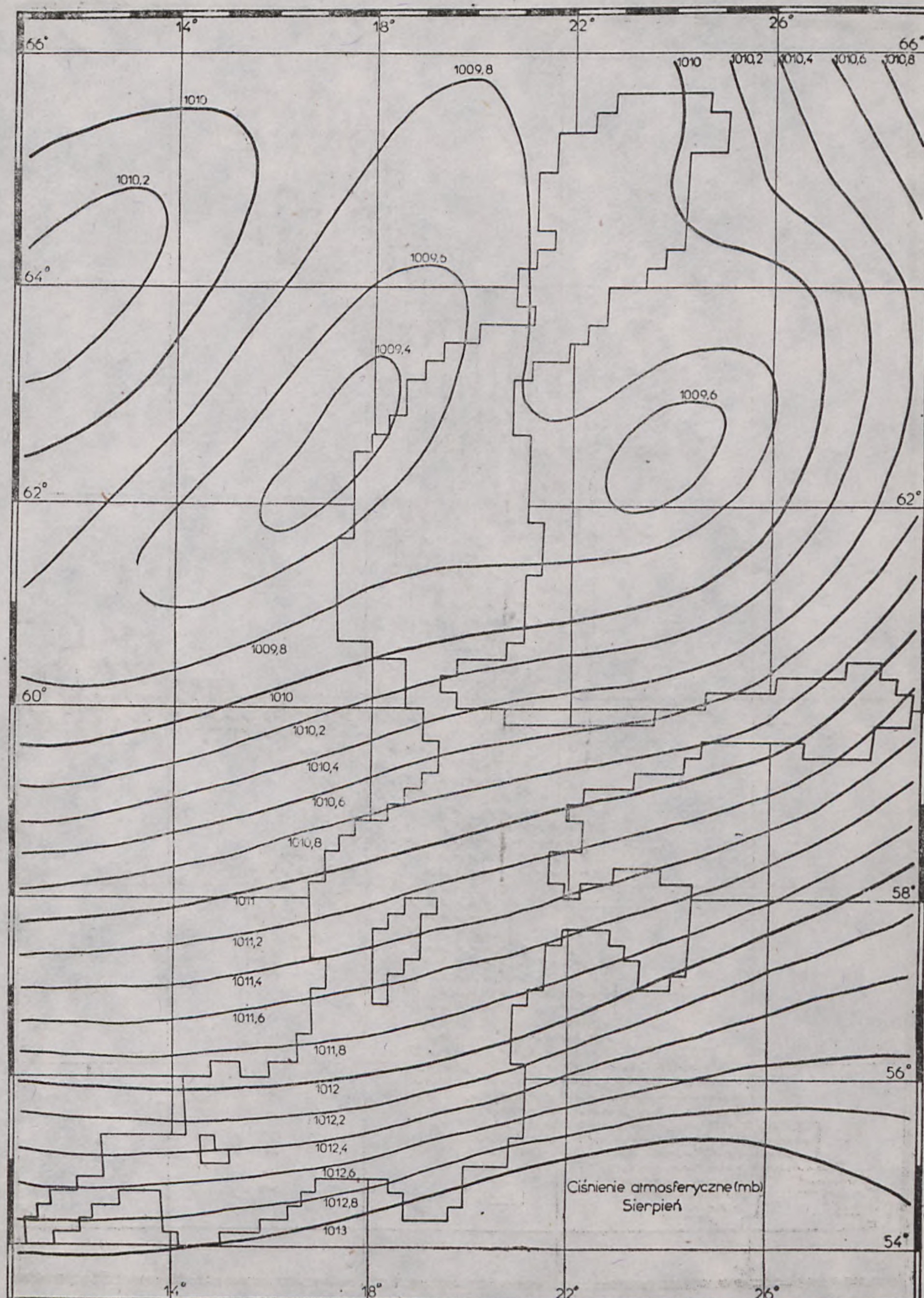

Fig. 7. Atmospheric pressure field [mb] for August (1951—1960).
Rys. 7. Pole ciśnienia atmosferycznego [mb] dla sierpnia (1951—1960) opracowane przez Oddział Morski IMGW w Gdyni.

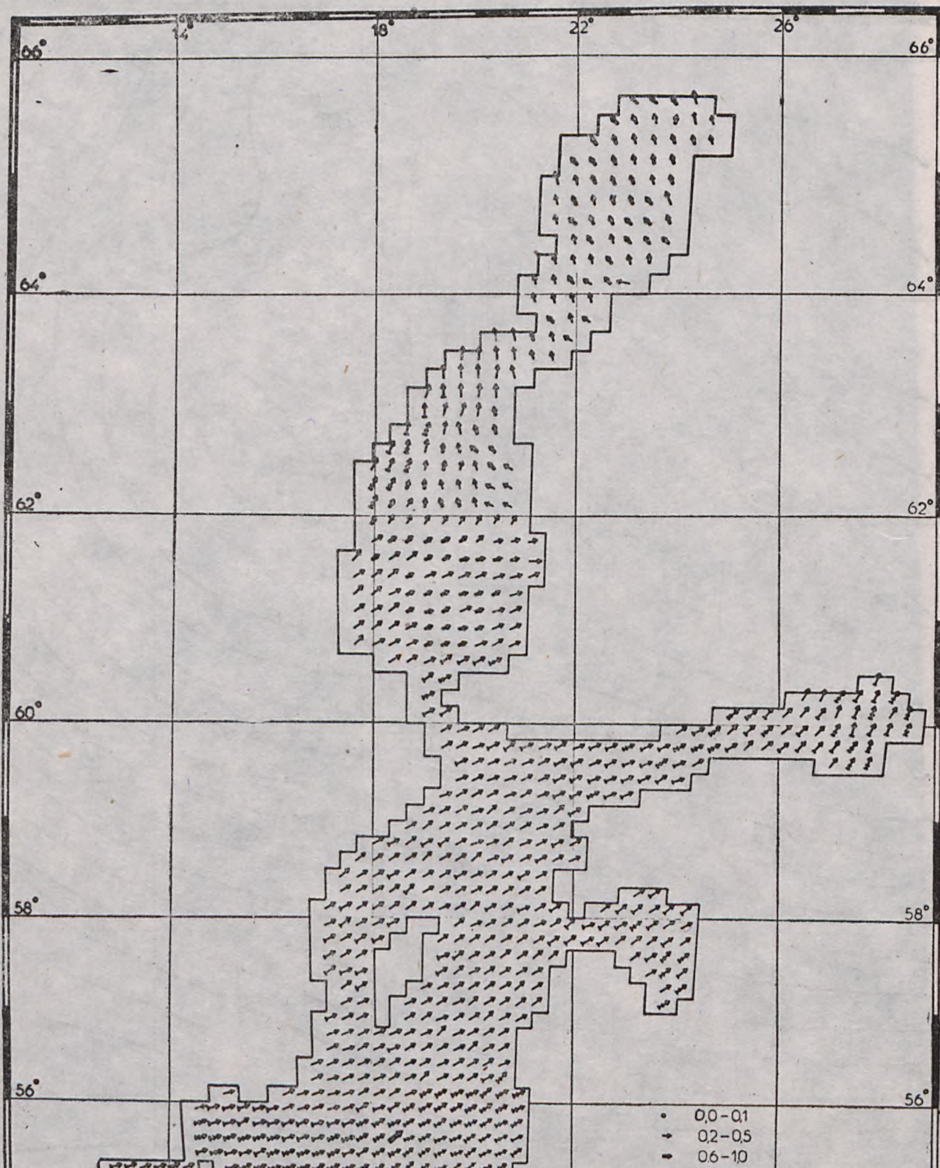

Fig. 8. Wind field [m s-1] for August.
Rys. 8. Pole wiatru [m s-1] dla sierpnia.

$$H\frac{\partial p_a}{\partial x} \quad ; \quad H\frac{\partial p_a}{\partial y} \qquad (77)$$

In the case of non-stationary circulation, spatial heterogeneity of the pressure field causes a specific configuration of the free sea surface, resulting in the occurrence of so-called barogradient currents [6]. If steady circulation is considered, the influence of spatial heterogeneity of the pressure field will appear as a corresponding configuration of the free sea surface. The H-N model used to investigate circulation in the Baltic Sea [10, 12] is based on a non-stationary system of equations (64—66) for mass transport and sea level with time constant generating forces (tangent wind stress and pressure). Integrating equations (64—66), after some time we obtain a steady state which is the result of energy dissipation due to the friction of liquid against the sea bottom. On approaching a steady state circulation, barogradient currents will be created, which should disappear in the course of time.

Taking into account the linearity of the system of equations for mass transport and sea level, the phenomenon of barogradient currents and its influence on the results of calculations of currents can be demonstrated. To attain this, let us consider the mass transport components M_x, M_y and the sea level ξ as the sum of dynamic components, generated directly by the tangent wind stress, and static components which are due to the heterogeneity of the atmospheric pressure field *:

$$M_x = M_x^c + M_x' \qquad (78)$$

$$M_y = M_y^c + M_y' \qquad (79)$$

$$\xi = \xi^c + \xi' \qquad (80)$$

where:
M_x^c, M_y^c, ξ^c — static components,
M_x', M_y', ξ' — dynamic components.
Substituting equations (78—80) into equation system (64—66) we obtain two systems of equations: one for dynamic quantities:

$$\frac{\partial M_x'}{\partial t} - \Omega M_y' = \tau_x - RM_x' - \varrho_0 gH\frac{\partial \xi'}{\partial x} \qquad (81)$$

$$\frac{\partial M_y'}{\partial t} + \Omega M_x' = \tau_y - RM_y' - \varrho_0 gH\frac{\partial \xi'}{\partial y} \qquad (82)$$

* This assumption is artificial, as in reality a heterogeneous field of atmospheric pressure immediately produces a wind.

$$\frac{\partial \xi'}{\partial t} = -\frac{1}{\varrho_0}\left(\frac{\partial M_x'}{\partial x} + \frac{\partial M_y'}{\partial y}\right) \tag{83}$$

and the second for static components:

$$\frac{\partial M_x^c}{\partial t} - \Omega M_y^c = -RM_x^c - \varrho_0 gH\frac{\partial \xi^c}{\partial x} - H\frac{\partial p_a}{\partial x} \tag{84}$$

$$\frac{\partial M_y^c}{\partial t} + \Omega M_x^c = -RM_y^c - \varrho_0 gH\frac{\partial \xi'}{\partial y} - H\frac{\partial p_a}{\partial y} \tag{85}$$

$$\frac{\partial \xi^c}{\partial t} = -\frac{1}{\varrho_0}\left(\frac{\partial M_x^c}{\partial x} + \frac{\partial M_y^c}{\partial y}\right) \tag{86}$$

Carrying out analogical considerations for current velocities, from formula (67) we obtain the expression for the calculation of static components of current:

$$D^c = -\frac{RM^c}{\varrho_0 p_1 A}\,\frac{\operatorname{ch} p_1 z}{\operatorname{sh} p_1 H} - \frac{1}{\varrho_0\Omega}\,i\left(\frac{\partial p_a}{\partial x} + i\frac{\partial p_a}{\partial y}\right) + \frac{g}{\Omega}\,i\left(\frac{\partial \xi^c}{\partial x} + i\frac{\partial \xi^c}{\partial y}\right) \tag{87}$$

where:

$$D^c = u^c + iv^c \quad , \quad M^c = M_x^c + iM_y^c \tag{88}$$

For steady circulation, i.e. when barogradient currents (static components) are equal to zero, we have:

$$u^c = v^c = 0 \qquad M_x^c = M_y^c = 0 \tag{89}$$

$$\frac{\partial p_a}{\partial x} = -\varrho_0 g\frac{\partial \xi^c}{\partial x} \tag{90}$$

$$\frac{\partial p_a}{\partial y} = -\varrho_0 g\frac{\partial \xi^c}{\partial y} \tag{91}$$

$$D^c = \frac{1}{\varrho_0\Omega}\,i\left(\frac{\partial p_a}{\partial x} + i\frac{\partial p_a}{\partial y}\right) + \frac{g}{\Omega}\,i\left(\frac{\partial \xi^c}{\partial x} + i\frac{\partial \xi^c}{\partial y}\right) = 0 + i0 \tag{92}$$

Equations (90) and (91) indicate that system of isolines of the sea level (static component) and that of isobars of the atmospheric pressure field, should be identical. Taking into account equations (84—86), also boundary and initial conditions (37) and (38) written for static components, we can calculate the barogradient circulation in the Baltic Sea.

The mean pressure field for August (Fig. 7) was taken as an exemplary atmospheric pressure field. Numerical calculations were carried out using the H-N numerical scheme [9, 10]) the spatial step h of the numerical grid being 5 nautical miles, and the time step $\Delta t = 60$ s. The attaining of a steady state of dynamic processes in the Baltic was observed on the example of the sea level. Exemplary oscillations of the free sea surface occurring during calculations at a given point in the Baltic basin, are illustrated in Fig. 9. The following expression was assumed as the criterion for the steady state of the level:

$$\Delta\xi = \max_{m,n} \left| \xi^{t+2\Delta t}_{m,n} - \xi^{t}_{m,n} \right| \leq \varepsilon_0 \qquad (93)$$

where:

m, n — indexes of the numerical grid,

$\xi^{t}_{m,n}\ \xi^{t+2\Delta t}_{m,n}$ — sea level ordinates at two consecutive time steps t and $t + 2\,\Delta\,t$, respectively.

Values of the parameters R and A were estimated according to relations (69—71) using the wind field for August (Fig. 8). The results of the calculations of mass transport, sea level and currents on the sea surface, are shown in Figs. 10—12.

The mass transport field (Fig. 10) is of a residual character. The values of mass transport are not high and do not exceed 1.6×10^3 g cm^{-1} s^{-1}. The sea level (Fig. 11) changes from -1.8 cm in the Gulf of Bothnia to 2.0 cm in Gdańsk Bay. The configuration of sea level isolines is very similar to the distribution of isobars of the atmospheric pressure field (Fig. 7). The system of current velocity vectors on the sea surface (Fig. 12) is similar to the isolines for the free sea surface (Fig. 11). This fact is justifiable as these are gradient currents [3]. The values of the currents attain 1.4 cm s^{-1}.

Let us discuss the results obtained. The sea level field calculated according to the above considerations, is related to the atmospheric pressure field; compare equations (90) and (91). The values of currents and transports are high compared with the expected values, which are close to zero. Let us now explain this discrepancy by the following example. Numerical calculations were carried out with an accuracy of $\varepsilon_0 = 0.005$ cm. Assume that at one of the nodes of the numerical grid, the calculated level ξ^c differs from that determined in reality, which was generated by a static pressure field $\xi^c{}_r$, with the value $\delta = 0.05$ cm:

$$\xi^c = \xi^c_r + \delta \qquad (94)$$

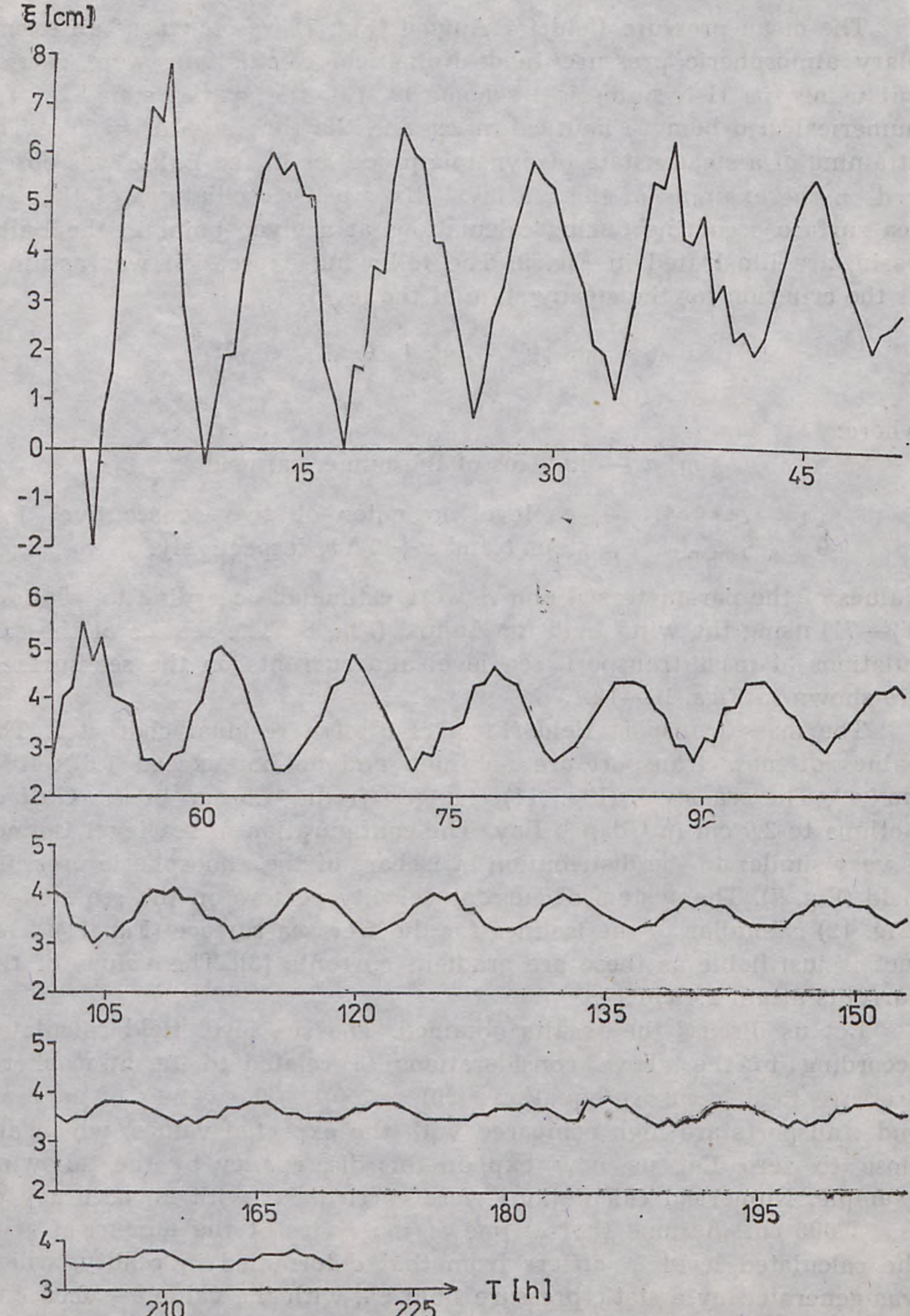

Fig. 9. Free sea surface variations during computations at a given point in the Baltic for barogradient circulation.

Rys. 9. Oscylacje swobodnej powierzchni morza w czasie obliczeń w wybranym punkcie Bałtyku dla przepływów barogradientowych.

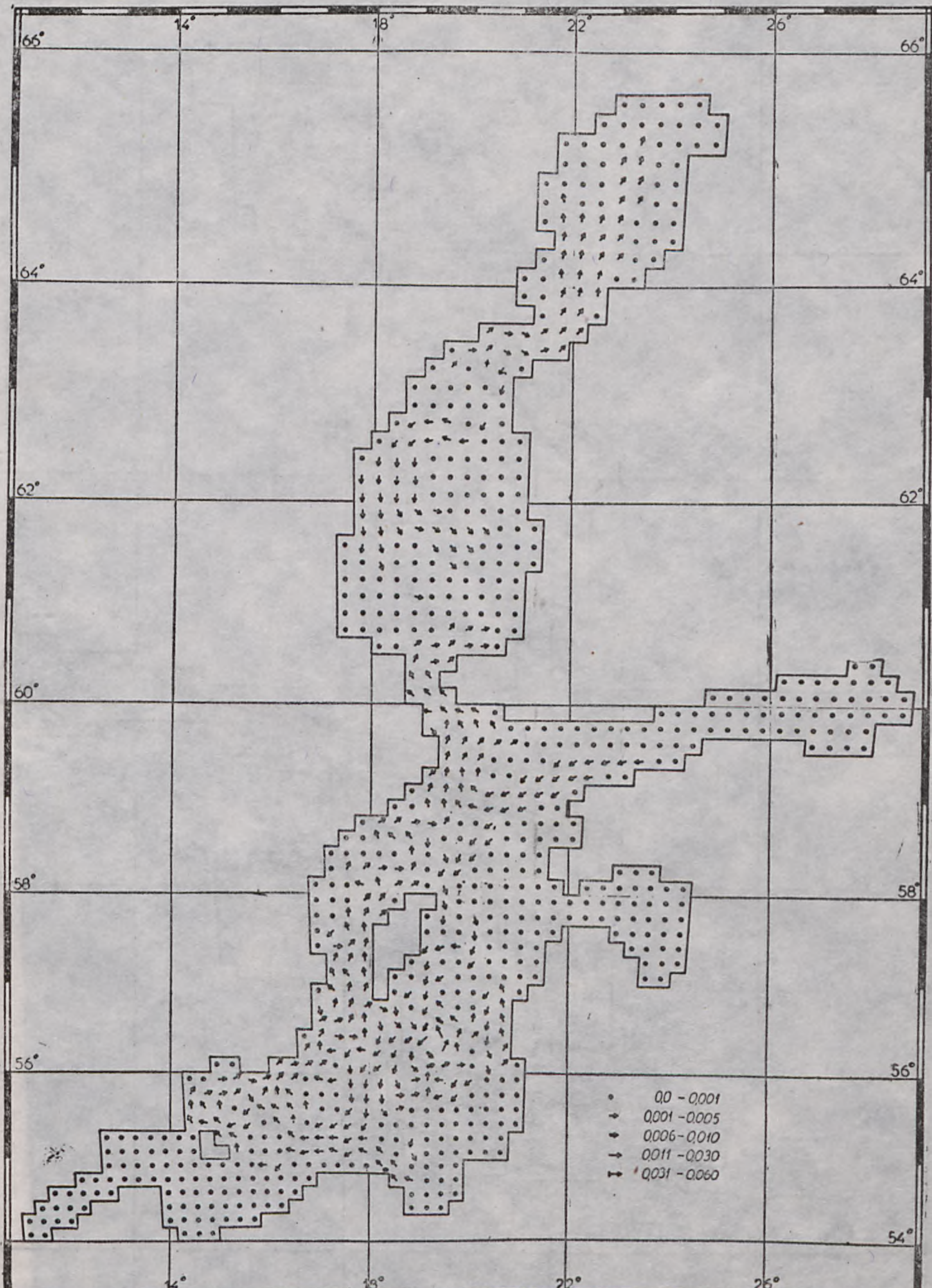

Fig. 10. Mass transport field [10^5 g cm^{-1} s^{-1}] in the Baltic Sea (homogeneous basin) for barogradient circulations.
Rys. 10. Pole wydatków masowych [10^5 g cm^{-1} s^{-1}] w Morzu Bałtyckim (akwen jednorodny) dla przepływów barogradientowych.

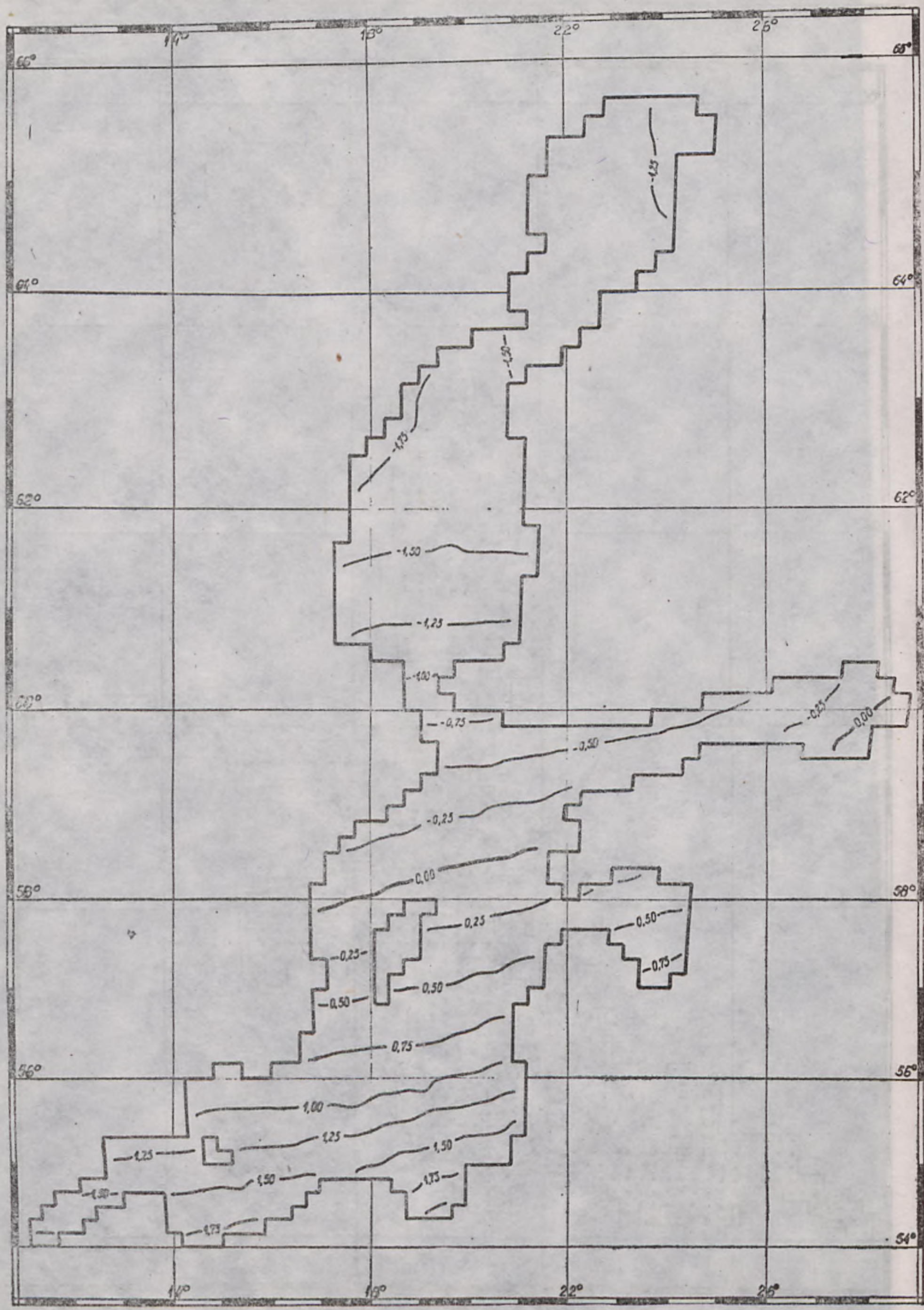

Fig. 11. Sea level field [cm] in the Baltic Sea (homogeneous basin) for barogradient circulations.
Rys. 11. Pole poziomu morza [cm] w Bałtyku (akwen jednorodny) dla przepływów barogradientowych.

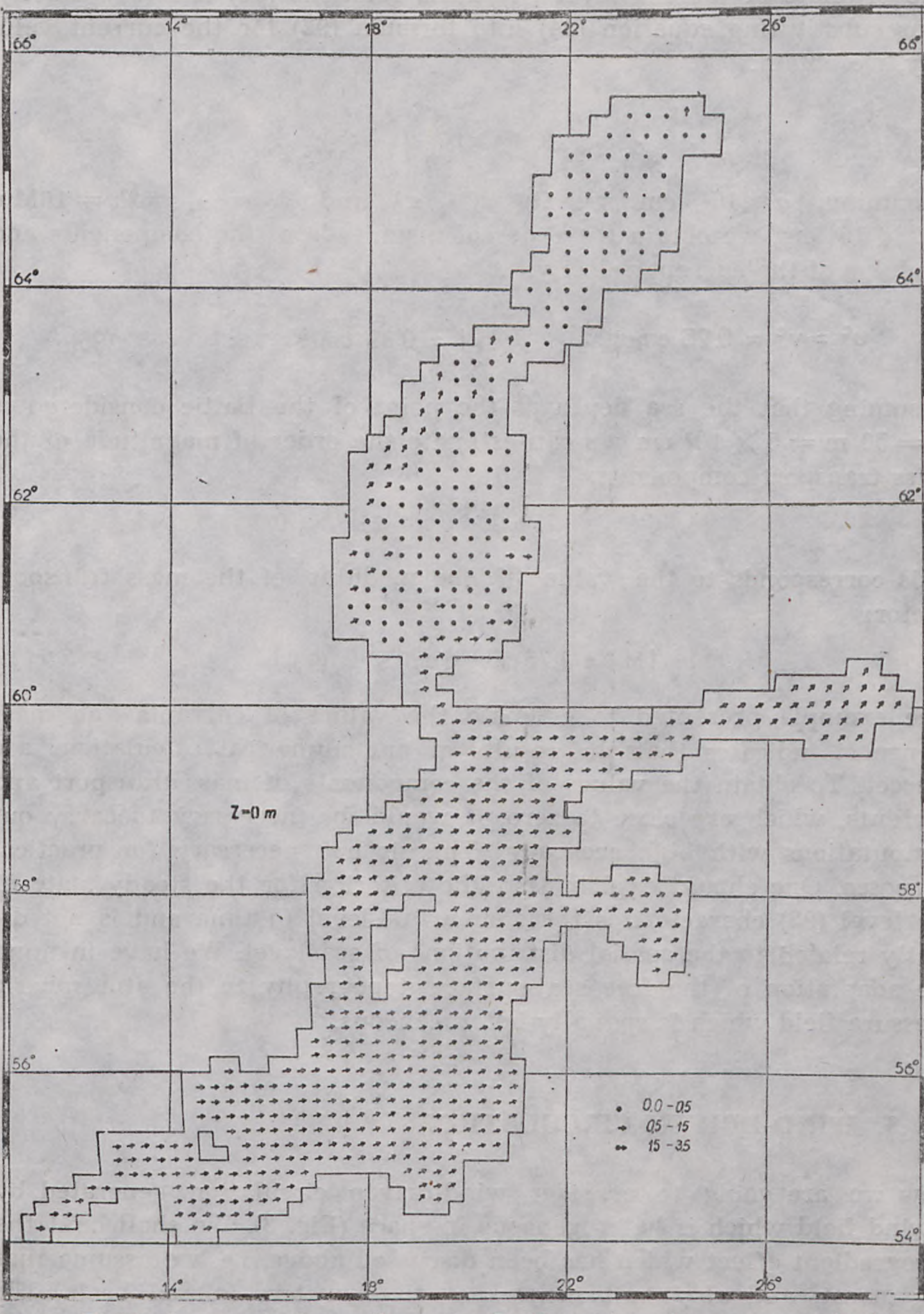

Fig. 12. Field of currants [cm s-1] on the Baltic surface (homogeneous basin) for barogradient circulations.
Rys. 12. Pole prądów [cm s-1] na powierzchni Morza Bałtyckiego (akwen jednorodny) dla przepływów barogradientowych.

The ordinate of the sea level ξ^c_r fulfils relations (90) and (91) exactly, thus substituting equation (94) into formula (92) for the current velocity, we obtain:

$$D^c = \frac{g}{\Omega} i\left(\frac{\delta}{\Delta x} + i\frac{\delta}{\Delta y}\right) \quad (95)$$

Assuming: $g \approx 10^3$ cm s^{-2}, $\Omega \approx 10^{-4}$ s^{-1} and $\Delta_x = \Delta_y = 2h = 10$Mm $\approx 2 \times 10^6$ cm, we obtain the order of magnitude of the components and modulus of the current velocity:

$$u^c = v^c \approx 0.25 \text{ cm s}^{-1} \qquad |\vec{v}| \approx 0.35 \text{ cm/s} \quad (96)$$

Assuming that the sea depth at the point of the Baltic considered is $H = 50$ m $= 5 \times 10^3$ cm we can estimate the order of magnitude of the mass transport components:

$$M_x = M_y \approx 1.25 \cdot 10^3 \text{ g cm}^{-1}\text{s}^{-1} \quad (97)$$

This corresponds to the value of the modulus of the mass transport vector:

$$|\vec{M}| \approx 1.76 \cdot 10^3 \text{ g cm}^{-1}\text{s}^{-1} \quad (98)$$

The example presented to estimate the values of currents and mass transport indicates that the results of our numerical calculations are correct. To obtain the values of the components of mass transport and currents, which are close to zero, it would be necessary to carry out computations with high accuracy. This is not necessary for practical purposes. One should explain that the criterion for the steady state of sea level (93) characterizes the changes of level in time and is not directly related to the spatial distributions of sea level. We have in mind the adaptation of the free sea surface topography to the atmospheric pressure field which is spatially heterogeneous.

8. WIND-DRIVEN CIRCULATION

Now we are going to consider wind-driven circulation generated by a wind field which is heterogeneous in space (Fig. 8). We shall omit the barogradient effect which has been discussed above, i.e. we assume that $p_a = \text{const}$, and also omit inflows of river waters (closed basin). To assess the influence of the wind field heterogeneity on the dynamics of the mass transport field, model computations will also be carried out for winds of constant velocity over the Baltic basin. Numerical computations were carried out using an H-N differential scheme [9, 10]. with a time step of 120 s.

Exemplary oscillations of the ordinate of free sea surface during calculations at a chosen point of the Baltic for heterogeneous wind are shown in Fig. 13. The mass transport and sea level field calculated for such wind are shown in Figs. 14 and 15.

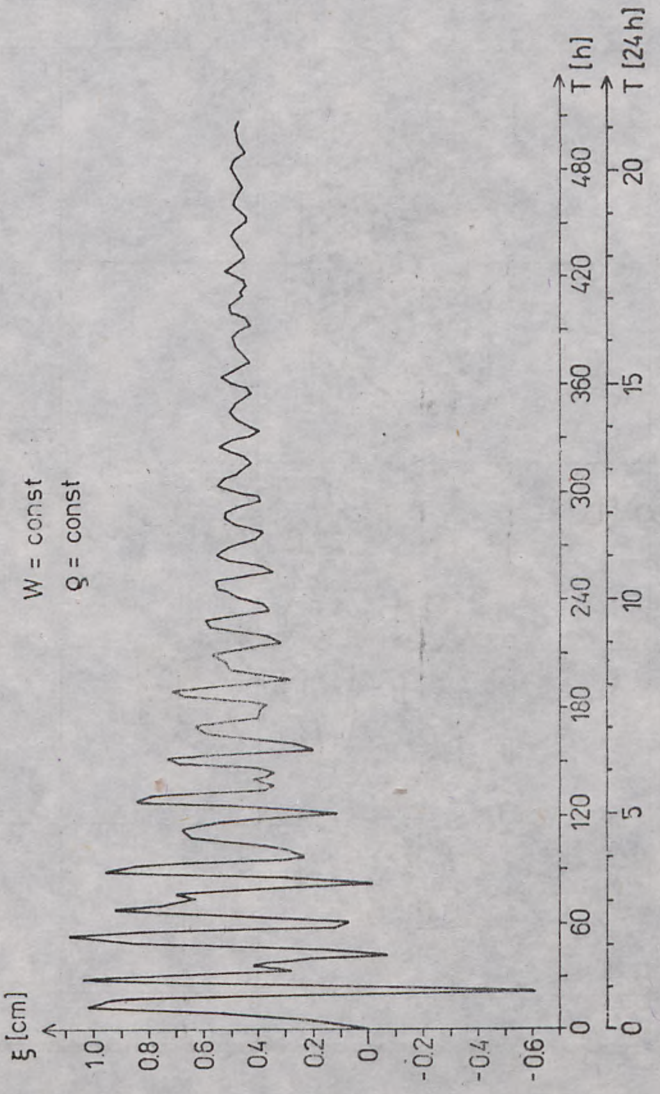

Fig. 13. Sea level variations during computations at a given point in the Baltic for heterogeneous wind (Fig. 8).

Rys. 13. Oscylacje swobodnej powierzchni morza w czasie obliczeń w wybranym punkcie Morza Bałtyckiego dla wiatru niejednorodnego (rys. 8).

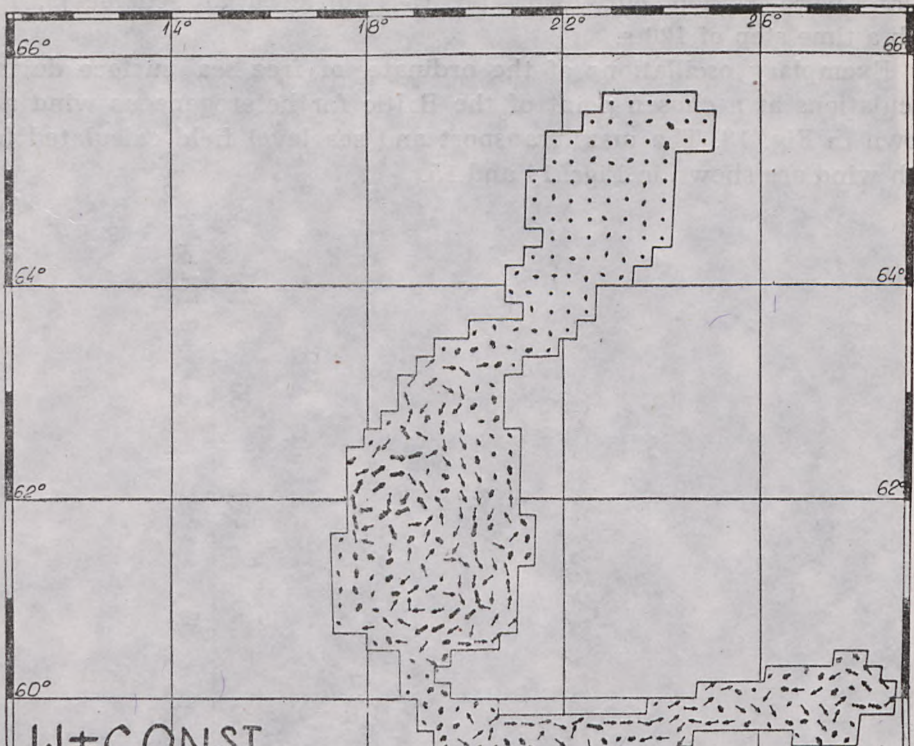

Fig. 14. Mass transport field [10^5 g cm^{-1} s^{-1}] in the Baltic (homogenous basin) for heterogeneous wind (Fig. 8).
Rys. 14. Pole wydatków masowych [10^5 g cm^{-1} s^{-1}] w Bałtyku (akwen jednorodny) dla wiatru niejednorodnego (rys. 8).

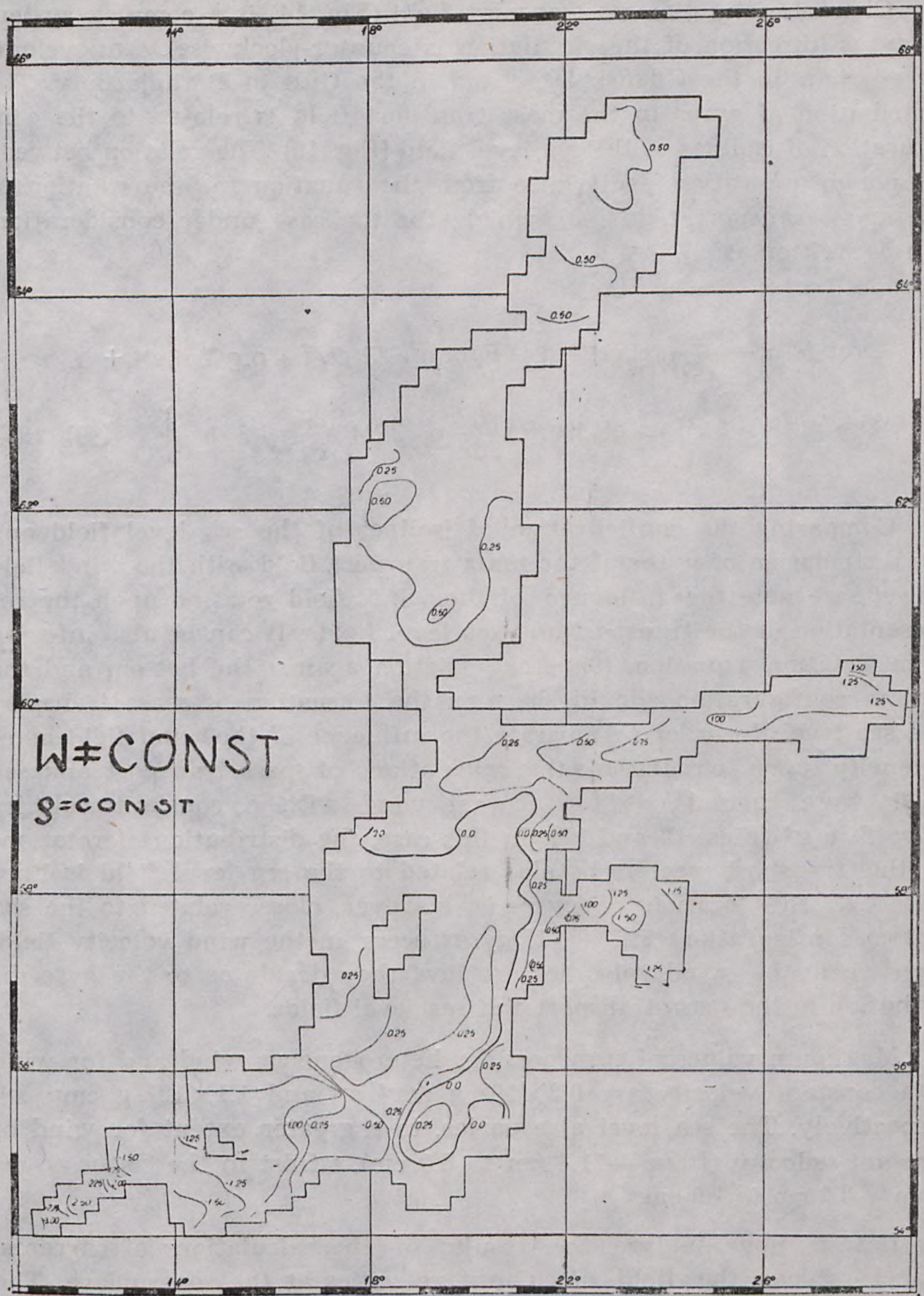

Fig. 15. Sea level field [cm] in the Baltic (homogeneous basin) for heterogeneous wind (Fig. 8).
Rys. 15. Pole poziomu morza [cm] w Bałtyku (akwen jednorodny) dla wiatru niejednorodnego (rys. 8).

Characteristic of mass transport field (Fig. 14) is a complex system of gyres (direction of the circulation is counter-clockwise). Anticyclonic gyres occur in the Gdańsk Deep and in the Gulf of Bothnia only. The distribution of gyres in the mass transport field is related to the configuration of isolines of the sea level field (Fig. 15). The relation between these two quantities results also from the equation for the rotation of the mass transport field (63), which, for the case under consideration, can be written as:

$$\mathrm{rot}_z \vec{M} = \frac{1}{R^2+\Omega^2}\Big\{\varrho_0 gH\Omega\Delta\xi + R\,\mathrm{rot}_z\vec{\tau} - \Omega\,\mathrm{div}\vec{\tau} + \varrho_0 g\Omega\nabla\xi\nabla H + $$
$$+ \varrho_0 gRJ(\xi,H) + \varrho_0 gHJ(\xi,R) + \frac{\partial R}{\partial x}(\tau_y + 2RM_y) - \frac{\partial R}{\partial y}(\tau_x - 2RM_x)\Big\} \qquad (99)$$

Comparing the configuration of isolines of the sea level field and the distribution of gyres of the mass transport field with the wind field (Fig. 8) we note the influence of the wind field rotation upon the representation of the transport and sea level fields. It can be also inferred from rotation equation (99) that friction against the bottom and the bottom configuration should share in the formation of mass transport and sea level. In order to separate the influence of the wind field heterogeneity upon circulation, the calculations of mass transport and sea level were carried out for a west wind with a constant velocity $W_a = 5$ m s^{-1} (Figs. 16 and 17). In this case, the distribution of rotations of the transport vectors is also related to the sea level field isolines (Fig. 17), the location of gyres is, however, closer related to the sea bottom configuration (Fig. 6). If gyres occur in the wind velocity field, even when the wind velocities are low, they decide as to the gyre distribution in the mass transport and sea level fields.

Maximum values of transport for heterogeneous wind and for wind of a constant velocity are 0.3×10^5 g cm^{-1} s^{-1} and 1.1×10^5 g cm^{-1} s^{-1}, respectively. The sea level also varies, to a greater extent for wind of constant velocity (from —11.4 cm to 8.0 cm), whilst in the former case, from —3.2 cm to 1.9 cm.

Let us now analyze the results of the calculations of currents. Fig. 18 shows the field of current velocities at the sea surface. The configuration of the vectors of currents is similar to that of the wind velocity vectors (Fig. 8). For a more detailed analysis of current fields, we classify the currents as drift and gradient, which we calculate according to the relation obtained from formula (67):

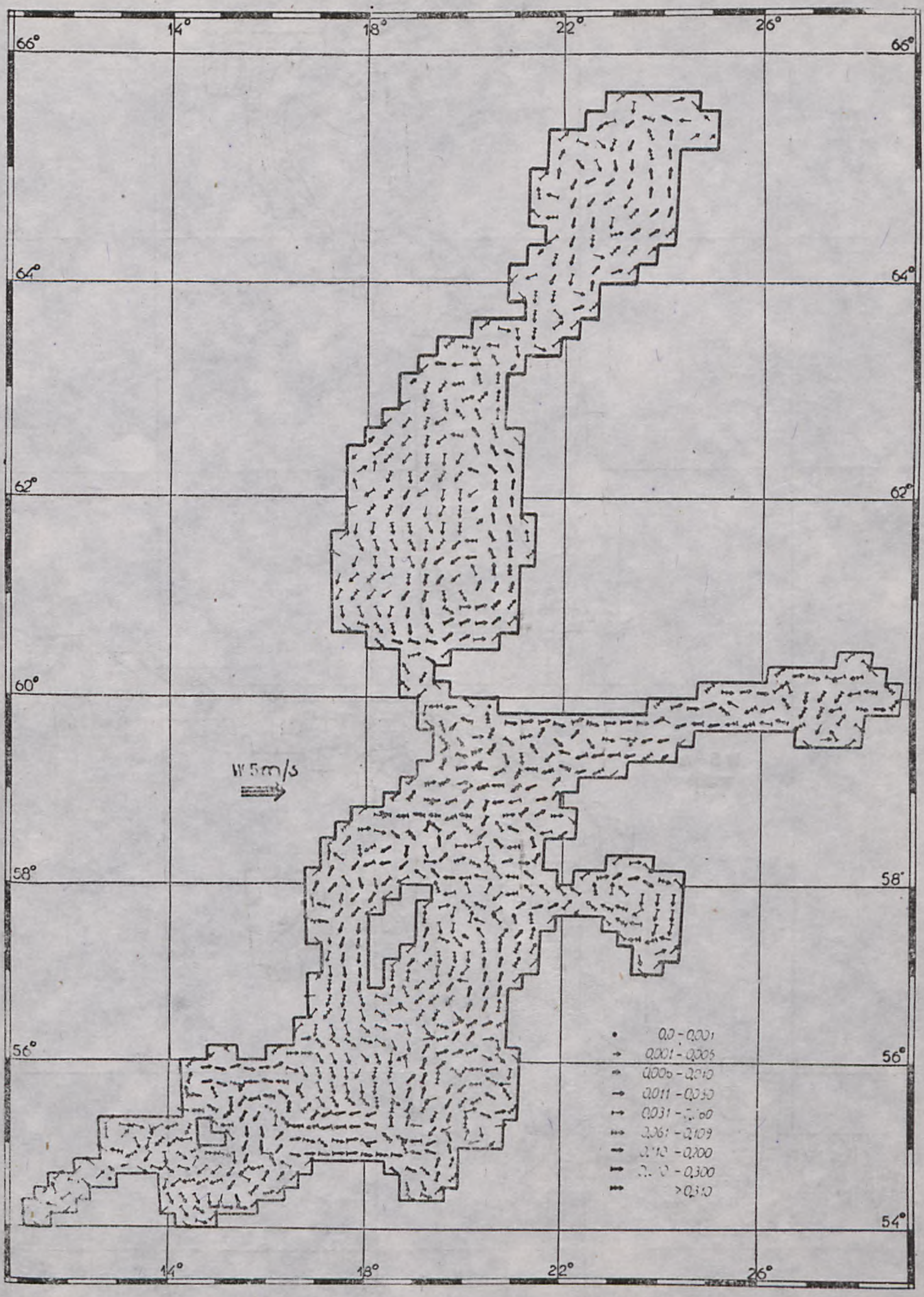

Fig. 16. Mass transport field [10^5 g cm^{-1} s^{-1}] in the Baltic (homogeneous basin) for a west wind of constant velocity $W_a = 5$ m s^{-1}.
Rys. 16. Pole wydatków masowych [10^5 g cm^{-1} s^{-1}] w Bałtyku (akwen jednorodny) dla wiatru zachodniego o stałej prędkości $W_a = 5$ m s^{-1}.

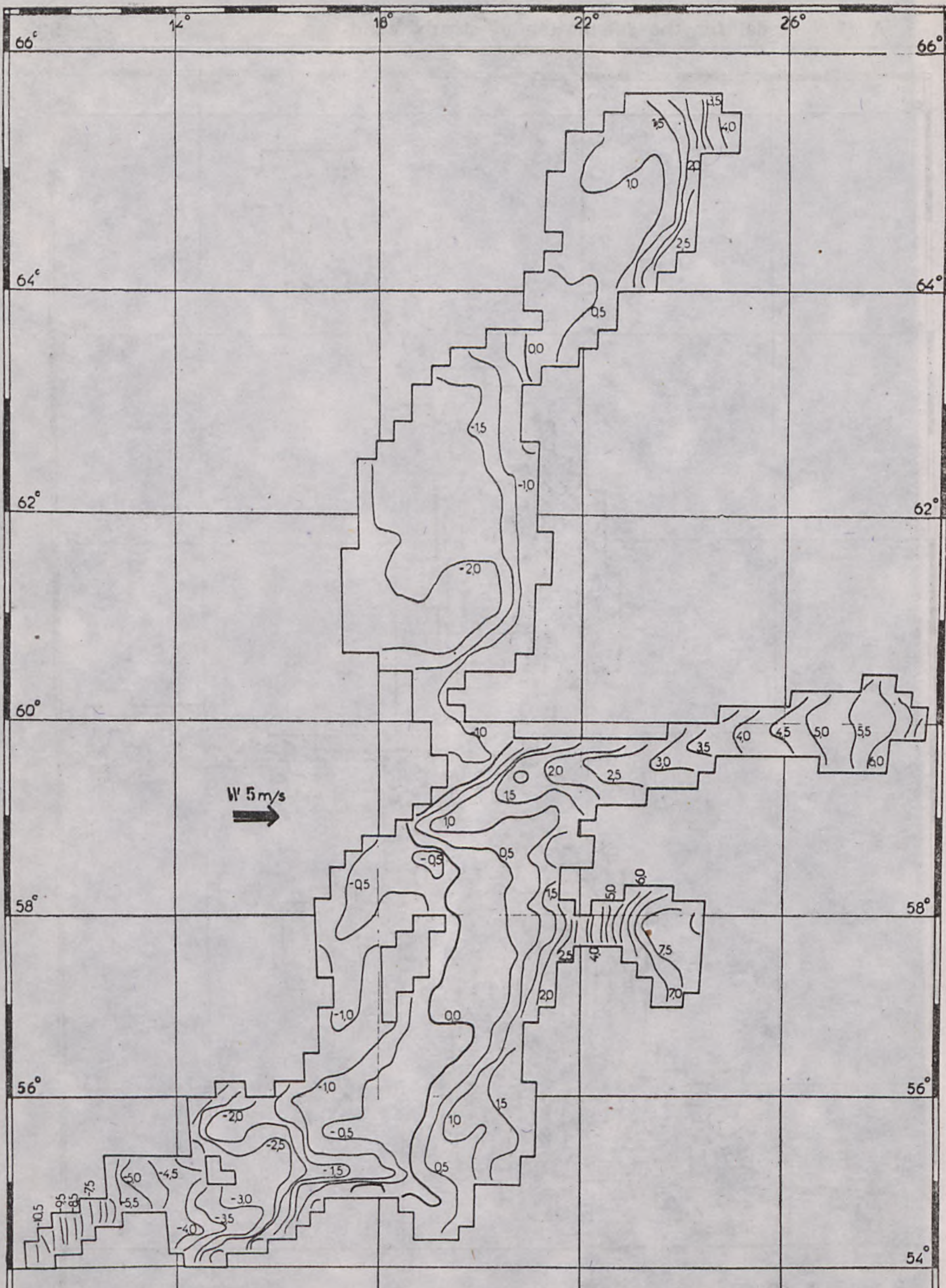

Fig. 17. Sea level field [cm] in the Baltic (homogeneous basin) for a west wind of constant velocity $W_a = 5$ m s^{-1}.
Rys. 17. Pole poziomu morza [cm] w Bałtyku (akwen jednorodny) dla wiatru zachodniego o stałej prędkości $W_a = 5$ m s^{-1}.

$$D_\tau = u_\tau + i v_\tau = \frac{\tau}{\varrho_0 p_1 A} \frac{\operatorname{ch} p_1(H+z)}{\operatorname{sh} p_1 H} - \frac{R M^\tau}{\varrho_0 p_1 A} \frac{\operatorname{ch} p_1 z}{\operatorname{sh} p_1 H} \tag{100}$$

$$D_g = u_g + i v_g = -\frac{R M^g}{\varrho_0 p_1 A} \frac{\operatorname{ch} p_1 z}{\operatorname{sh} p_1 H} + \frac{g}{\Omega} i \left(\frac{\partial \xi}{\partial x} + i \frac{\partial \xi}{\partial y} \right) \tag{101}$$

where:

$$M^\tau = \frac{1}{R^2 + \Omega^2} \left[R\tau_x + \Omega\tau_y + i\,(R\tau_y - \Omega\tau_x) \right] \tag{102}$$

$$M^g = \frac{1}{R^2 + \Omega^2} \left[R \frac{\partial \xi}{\partial x} + \Omega \frac{\partial \xi}{\partial y} + i \left(R \frac{\partial \xi}{\partial y} - \Omega \frac{\partial \xi}{\partial x} \right) \right] \tag{103}$$

denote the drift and gradient part of the complex transport.

Fields of drift and gradient currents on the sea surface are shown in Figs. 19 and 20. Analyzing Fig. 19 one can see the similarity between the scheme of vectors of drift current and those of wind velocity (Fig. 8). Gradient currents (Fig. 20) attain fairly low velocities, the distribution of the vectors being correlated with the configuration of sea level isolines (Fig. 15). The spatial image of the field of wind-driven currents can be supplemented by vertical distributions of the components of the velocities of full, drift and gradient currents at chosen points in the Baltic (Figs. 21 and 22). These figures also give the values of the Ekman friction depth, D_E, which were calculated according to the analytical relation:

$$D_E = \pi \sqrt{\frac{2A}{\Omega}} \tag{104}$$

also D^* estimated by means of the following relation [3]:

$$v^\tau_{D^*} \cong 0.043\, v^\tau_0 \tag{105}$$

where:

v^τ_0 — modulus of the drift current velocity at the sea surface,

$v^\tau_{D^*}$ — modulus of the drift current velocity at depth D^*.

The values of D_E and D^* are very close. The analysis of Figs. 21 and 22 indicates that the drift currents disappear rapidly with the increasing depth. Gradient currents reach the bottom and do not exhibit any changes except for a thin layer near the bottom where they undergo a slight transformation.

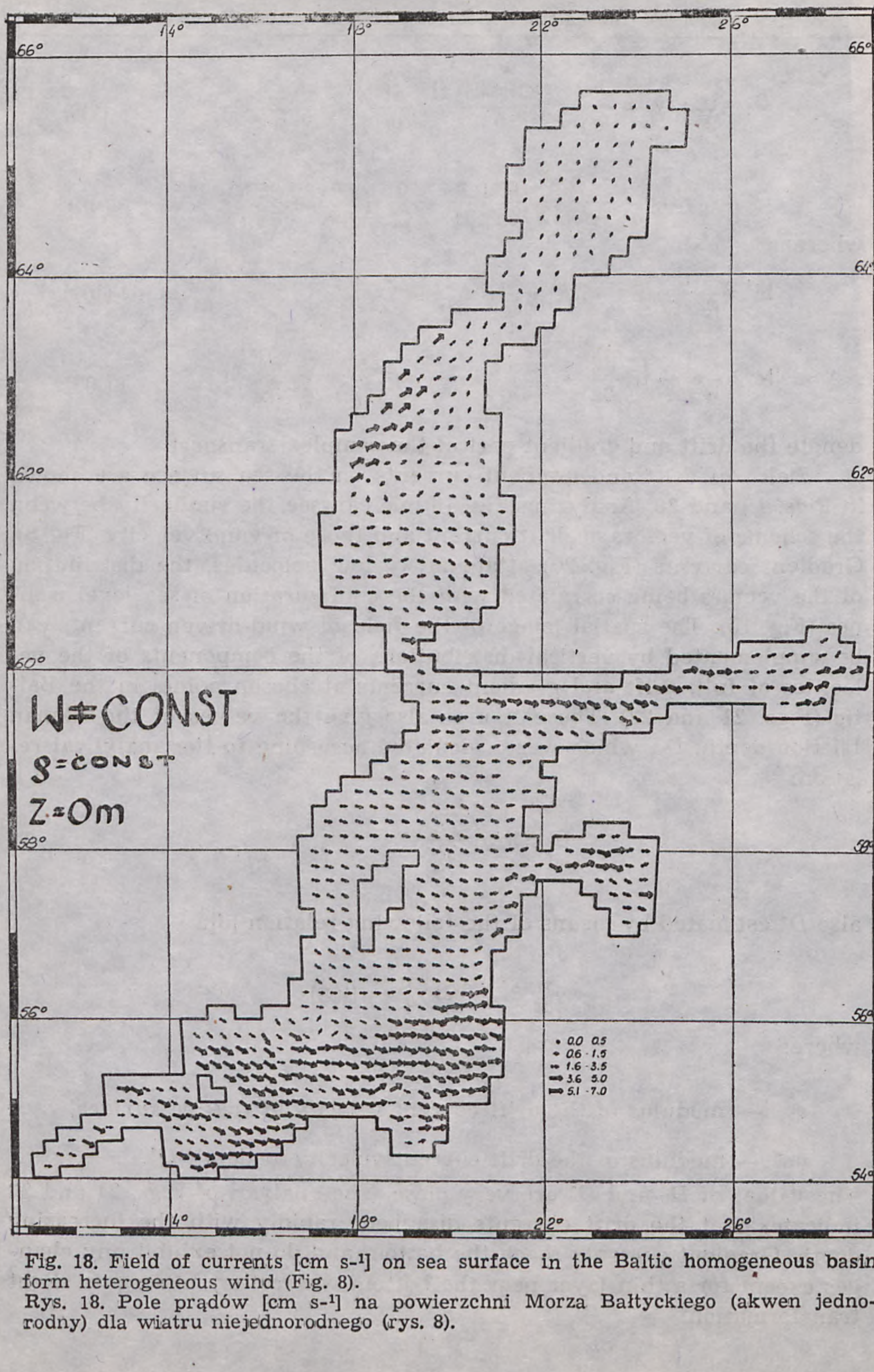

Fig. 18. Field of currents [cm s-1] on sea surface in the Baltic homogeneous basin form heterogeneous wind (Fig. 8).
Rys. 18. Pole prądów [cm s-1] na powierzchni Morza Bałtyckiego (akwen jednorodny) dla wiatru niejednorodnego (rys. 8).

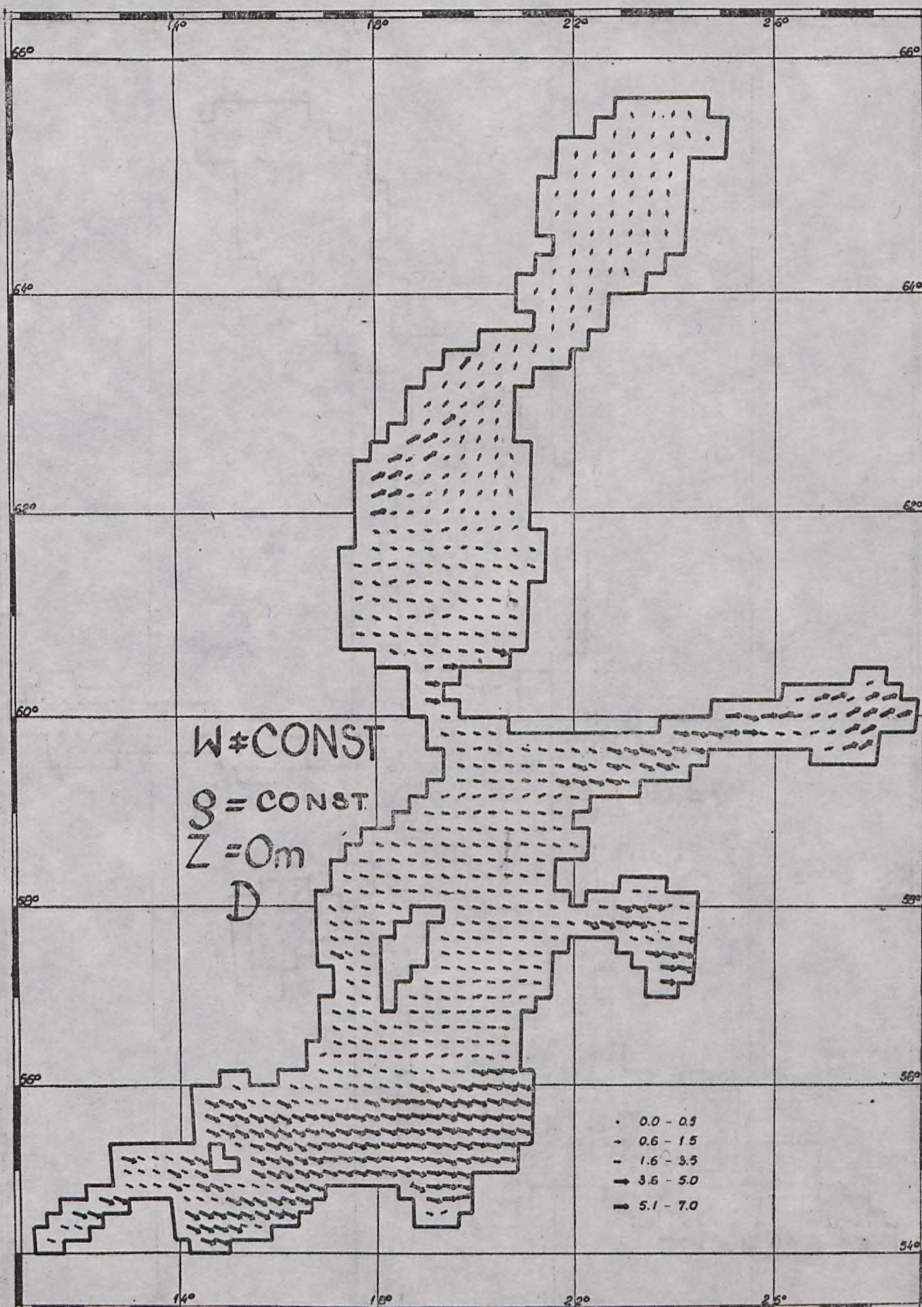

Fig. 19. Field of drift currents [cm s-1] on sea surface in the Baltic (homogeneous basin) for heterogeneous wind (Fig. 8).
Rys. 19. Pole prądów dryfowych [cm s-1] na powierzchni Bałtyku (akwen jednorodny) dla wiatru niejednorodnego (rys. 8).

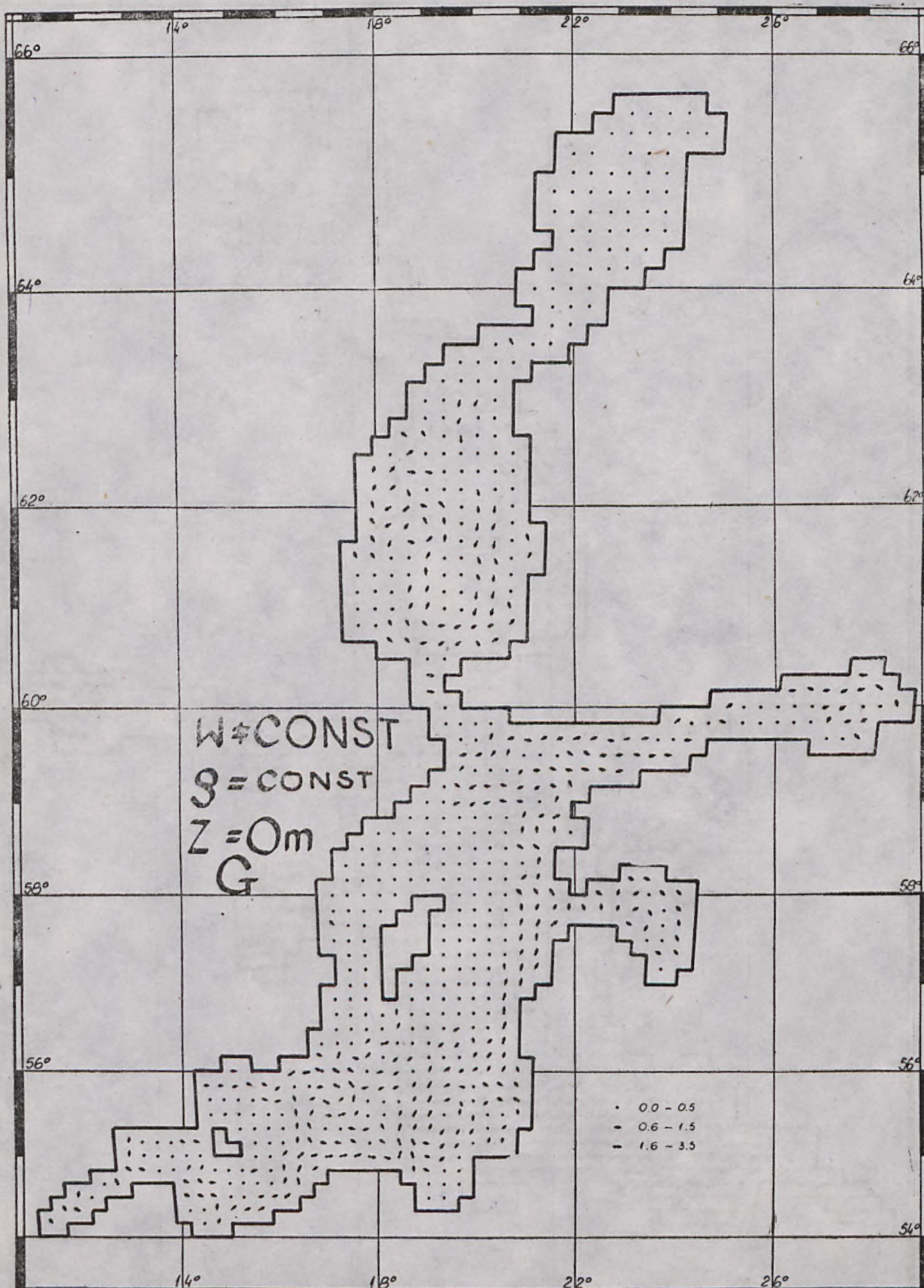

Fig. 20. Field of gradient currents [cm s-1] on sea surface in Baltic (homogeneous basin) for heterogeneous wind (Fig. 8).
Rys. 20. Pole prądów gradientowych [cm s-1] na powierzchni Bałtyku (akwen jednorodny) dla wiatru niejednorodnego (rys. 8).

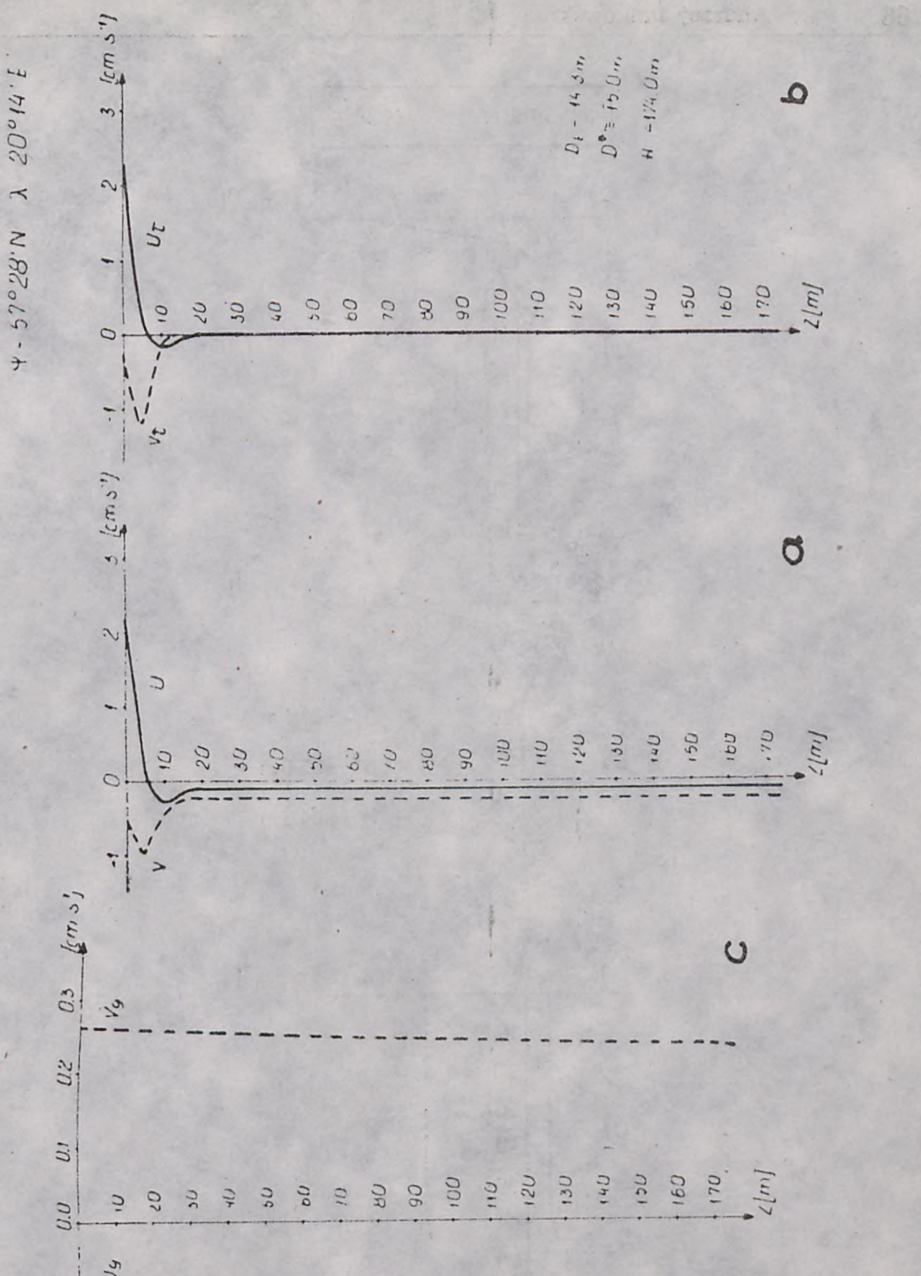

Fig. 21. Vertical structure of the current velocity [cm s^{-1}] in the Baltic (homogeneous basin, heterogeneous wind field) at point: $\varphi = 57°28'$ N, $\lambda = 20°14'$ E: a) total components, b) drift components, c) gradient components.

Rys. 21. Struktura pionowa prędkości prądu [cm s^{-1}] w Bałtyku (akwen jednorodny, niejednorodne pole wiatru) w punkcie o współrzędnych $\varphi = 57°28'$ N, $\lambda = 20°14'$ E: a) składowe pełne, b) składowe dryfowe, c) składowe gradientowe.

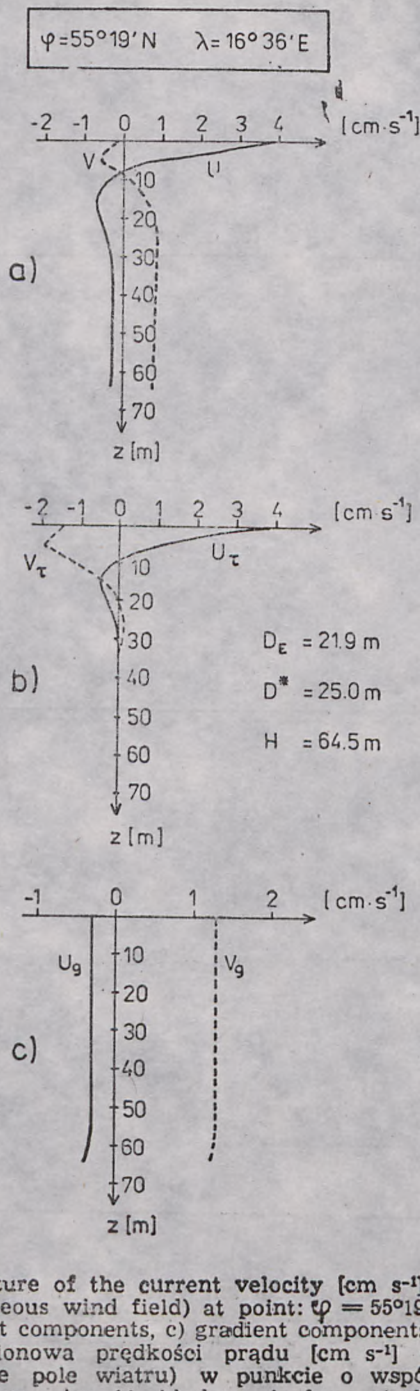

Fig. 22. Vertical structure of the current velocity [cm s^{-1}] in the Baltic (homogeneous basin, heterogeneous wind field) at point: $\varphi = 55°19'$ N, $\lambda = 16°36'$ E: a) total components, b) droft components, c) gradient components.
Rys. 22. Struktura pionowa prędkości prądu [cm s^{-1}] w Bałtyku (akwen jednorodny, niejednorodne pole wiatru) w punkcie o współrzędnych $\varphi = 55°19'$ N, $\gamma = 16°36'$ E: a) składowe pełne, b) składowe dryfowe, c) składowe gradientowe.

9. CIRCULATION DUE TO THE INFLOW OF RIVERS INTO THE BALTIC

The linearity of the H-N model enables the estimation of the water circulation generated by the inflow of river waters into the Baltic basin. Assuming the components of tangent wind stress in equations (64—65) to be equal to zero ($\tau_x = \tau_y = 0$) and $p_a = \text{const}$, and assuming the river inflows in the boundary condition (38), we obtain a system of equations suitable for the estimation of river circulation in the Baltic Sea. The values of river inflows (mean values for August in 1961—1970) worked out by Mikulski [19] (Table 2) were used in the calculations. The results of numerical calculations are illustrated in the maps of mass transport, sea level and currents at the sea surface (Figs. 23—25).

The mass transport field (Fig. 23) is characterized by the specific configuration of the transport vectors. Water masses carried in by the rivers, travel from the river mouths towards the Danish Straits along the isobaths of the depth field (Fig. 6), under the influence of gravitational and Coriolis forces. The values of the mass transport are low and do not exceed 3×10^3 g cm^{-1} s^{-1}.

The distribution of the sea level isolines (Fig. 24) and the vectors of currents at the sea surface (Fig. 25) are similar to the vectors of mass transport. Maximum values of current velocities occur in the vicinity of the Danish Straits and amount to 2.5 cm s^{-1}. Extreme values of the sea level also occur in this region and are of the order of —1.4 cm.

10. CONCLUSIONS

The results of the calculations of wind-driven circulation in a homogeneous basin indicate that the H-N model employed yields results which are qualitatively consistent with the Ekman theory of wind-driven currents (compare e.g. [3]). The relation of the rotation of the mass transport vector field to the sea level and wind fields, and to the sea bottom configuration, obtained by theoretical considerations (63, 69), was confirmed during the analysis of the results of the calculations.

In the author's opinion, the H-N model can be successfully employed for model investigations of wind-driven circulation in basins of medium depths. The weak point of the investigations lies in the manner of estimating the wind velocity above the Baltic (heterogeneous wind field in August — Fig. 8). Investigations by Defant [2] and Taranowska [27] indicate, that the wind velocities above the Baltic Sea in summer should be of the order of 6—7 m s^{-1}. It can therefore be concluded that the values of coefficient B in formula (76) are umderrated

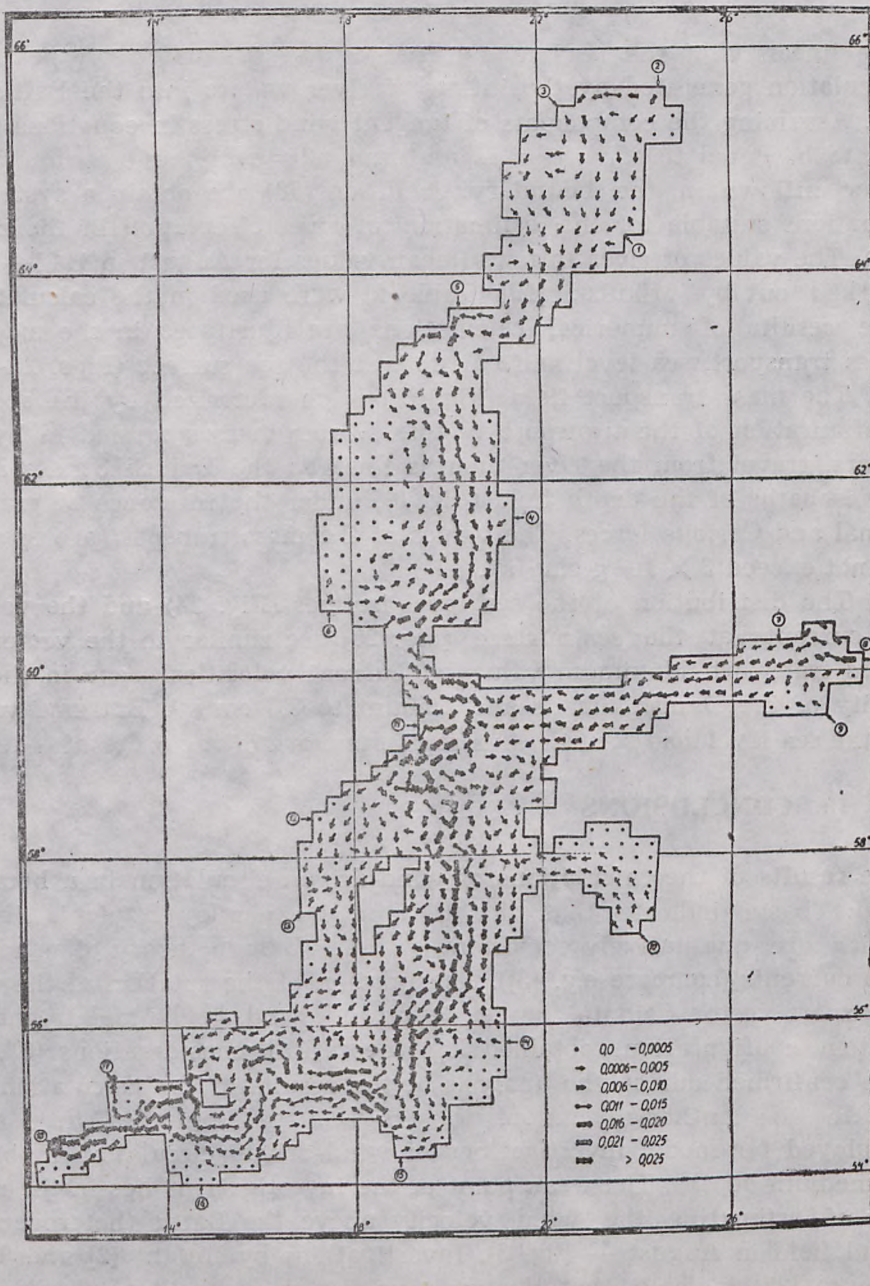

Fig. 23. Mass transport field [10^5 g cm^{-1}] in the Baltic (homogeneous basin) for river circulation. Arrows with numbers indicate the mouths of Baltic rivers.
Rys. 23. Pole wydatków masowych [10^5 g cm^{-1} s^{-1}] w Bałtyku (akwen jednorodny) dla cyrkulacji rzecznej. Strzałki z liczbami oznaczają ujścia rzek bałtyckich.

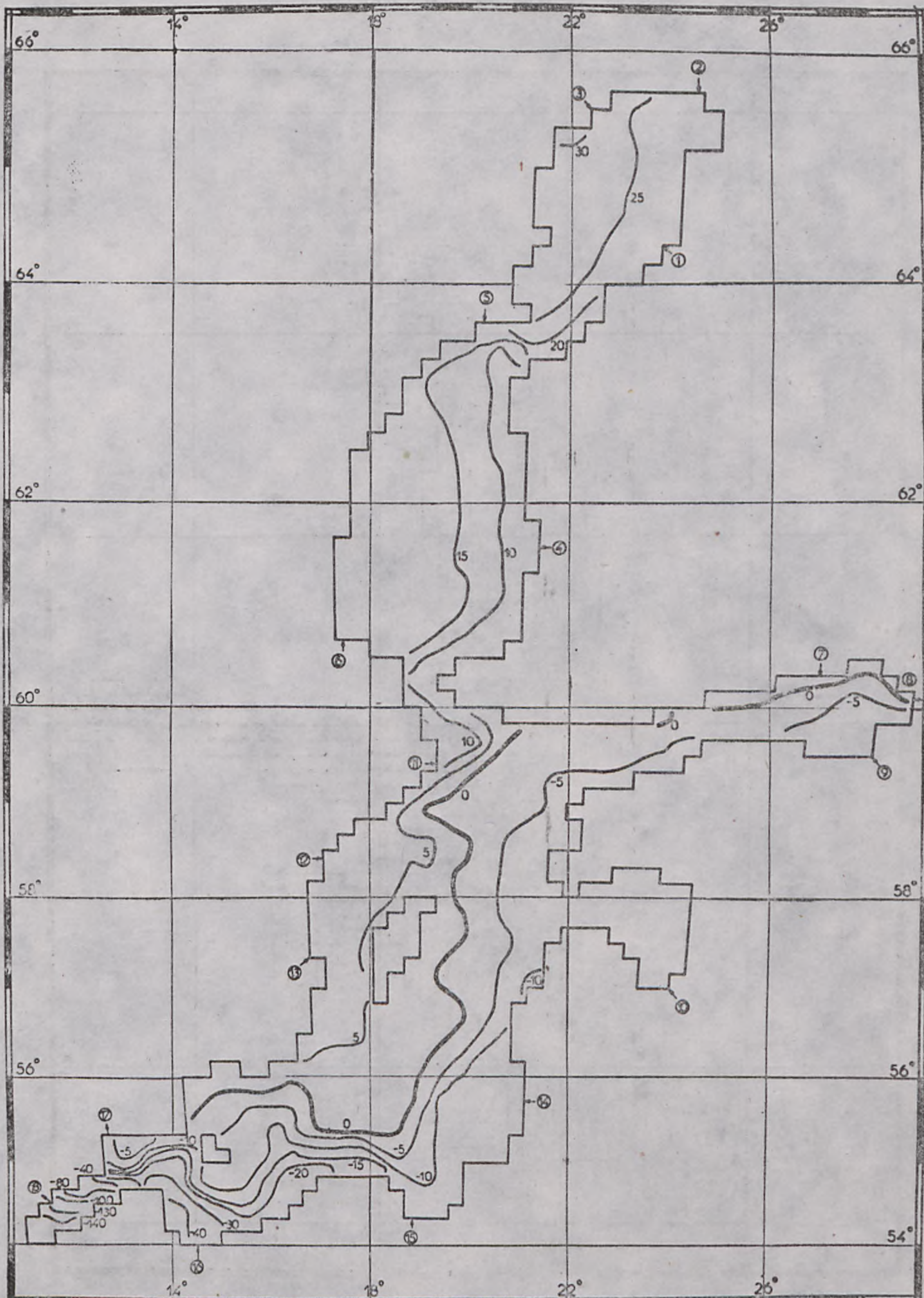

Fig. 24. Field of the sea level [cm] in the Baltic (homogeneous basin) for river circulation. Arrows with numbers indicate the mouths of rivers.
Rys. 24. Pole poziomu morza [cm] w Bałtyku (akwen jednorodny) dla cyrkulacji rzecznej. Strzałki z liczbami oznaczają ujścia rzek.

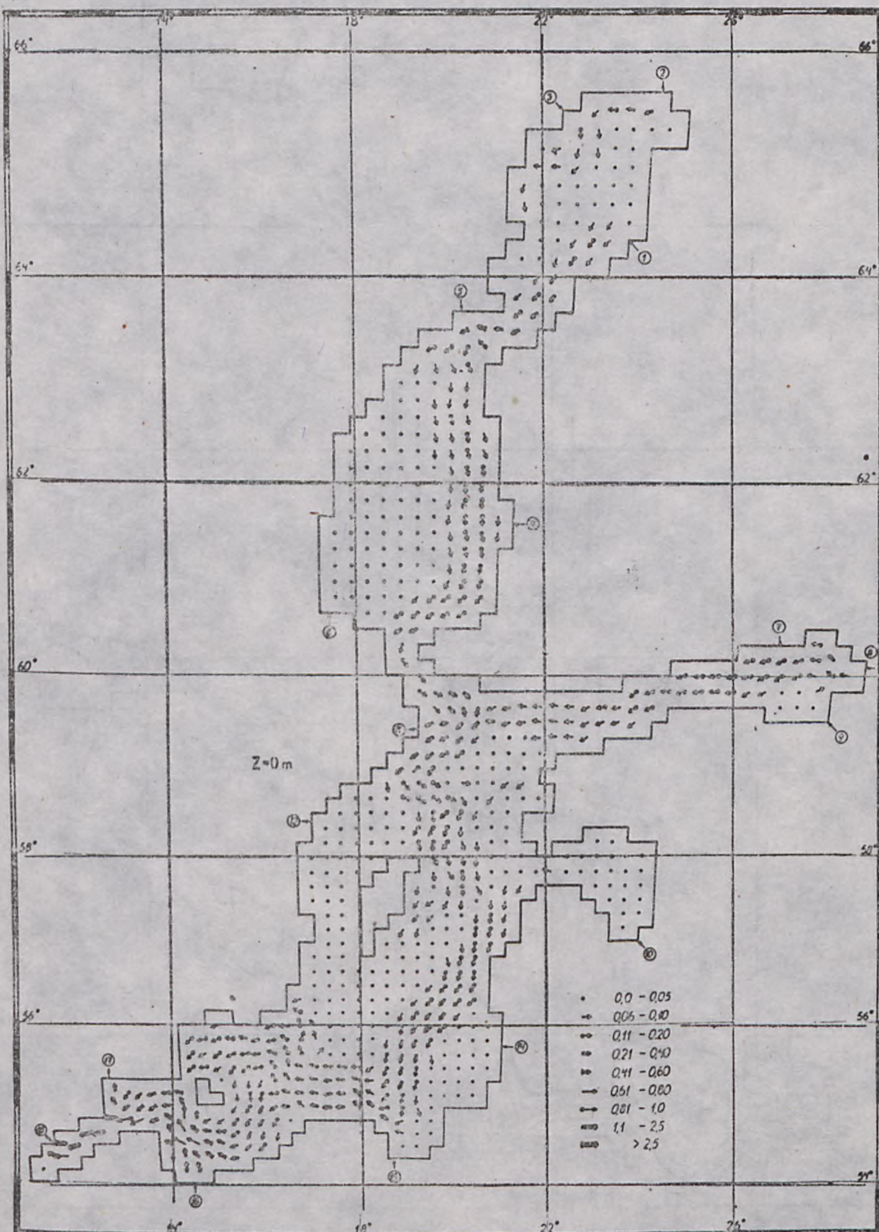

Fig. 25. Field of current [cm s^{-1}] on the sea surface in the Baltic (homogeneous basin) for river circulation. Arrows with numbers indicate the mouths of rivers.
Rys. 25. Pole prądów [cm s^{-1}] na powierzchni Bałtyku (akwen jednorodny) dla cyrkulacji rzecznej. Strzałki z liczbami oznaczają ujścia rzek.

and a correction should be made when the wind velocity is calculated basing on mean fields of atmospheric pressure obtained over a period of many years.

Comparing the results of the calculations of wind-driven and river circulations, it can be maintained that the role of rivers in the formation of the mass transport field, the sea level and currents is insignificant. The circulation caused by river inflows reach higher values only in the vicinity of river estuaries and in the region of the Danish Straits, where the water flows into the North Sea. There, there might be a modification of wind-driven circulation due to the river inflow. These effects will, however, be weak, and noticeable only in cases of winds of low velocity. Hence, it can be inferred that the variations in horizontal movement of water masses, due to the inflow of river waters into the Baltic Sea, are of purely local character. Modelling the wind-driven circulation in the whole Baltic one can omit river inflows, particularly in the case of strong winds.

The barogradient effect presented in this paper served to assess the correctness of the method of calculating stationary circulation using a non-stationary system of equations. The influence of the spatial heterogeneity of the atmospheric pressure field appeared only with a corresponding configuration of the sea level field. The fields of mass transport and currents result from the dynamic interaction of the wind field (tangent stress). Static distribution of the sea level in the case of a closed basin, can be estimated by means of equations (90, 91) with the additional condition, i. e. the maintaining of the volume of water in the basin [6]:

$$\iint_S \xi^c \, dx \, dy \qquad (106)$$

where:

S — the surface of the basin.

Setting the value of the sea level ordinate at any point of the basin and using equations (90, 91, 106) we obtain the static topography of the free sea surface, which is generated by a spatially heterogeneous field of atmospheric pressure.

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MODEL HYDRODYNAMICZNO-NUMERYCZNY DO OBLICZEŃ USTALONYCH PRZEPŁYWÓW WIATROWYCH I GĘSTOŚCIOWYCH W MORZU BAŁTYCKIM

Streszczenie

Omówione są podstawy teoretyczne modelu hydrodynamiczno-numerycznego H-N i przedstawione wyniki obliczeń przepływów wiatrowych dla sierpnia w jednorodnym akwenie bałtyckim. Ustalone przepływy gęstościowe dla tego okresu będą przedmiotem drugiej części opracowania [11].

Model H-N bazuje na nieustalonym, liniowym układzie równań ruchu ze stałymi, w czasie, siłami wymuszającymi (naprężenie styczne wiatru i naprężenie wywołane niejednorodnością gęstości wody) (16, 17) i równaniu ciągłości (4). Całkując wymienione równania od dna do powierzchni morza i uwzględniając warunki brzegowe (6 — 9) otrzymano układ równań dla wydatków masowych i poziomu morza (32 — 34). Wykorzystując ustaloną postać równań ruchu (16, 17) oraz warunki brzegowe (6, 8) wyprowadzono analityczne wyrażenie do obliczenia składowych prędkości prądu (23).

W dalszej części pracy wykazano jednoznaczność rozwiązania równań wydatków masowych i poziomu morza (32 — 34) przy zadanych warunkach początkowych (37) i brzegowych (38). Wyprowadzono również wyrażenia do oceny składowej pionowej wirowości pola wektorów wydatków masowych (63).

Model H-N bazuje na równaniu hydrostatyki, co eliminuje z rozważań przepływy o dużych prędkościach i przyspieszeniach pionowych. Bałtyk charakteryzuje się dużą zmiennością reliefu dna morza [5, 25]. Przy numerycznym rozwiązywaniu zagadnienia przepływów może to być przyczyną generacji silnych przepływów pionowych, których model nie opisuje. Może się to stać przyczyną niestabilności rozwiązania numerycznego. Aby tego uniknąć, dokonano wygładzenia reliefu dna Morza Bałtyckiego za pomocą wyrażenia (68), usuwając w ten sposób gwałtowne zmiany głębokości przy przejściu od jednego węzła siatki numerycznej do drugiego. Tak opracowaną batymetrię Bałtyku przedstawia rys. 6.

Obliczenia przepływów wiatrowych w jednorodnym Bałtyku przeprowadzono dla średniego pola wiatru dla sierpnia (rys. 8). Rysunki: 14, 15, 18 — 20 ilustrują mapy wydatków masowych, poziomu morza oraz prądów pełnych, dryfowych i gradientowych na powierzchni Bałtyku. Strukturę przestrzenną prądów dopełniają rozkłady składowych pełnych, dryfowych i gradientowych w wybranych punktach Bałtyku. (rys. 21, 22). W celu oceny wpływu niejednorodności pola wiatru na pola wydatków masowych i poziomu morza wykonano również obliczenia dla wiatrów zachodnich o stałej prędkości 5 m s^{-1} (rys. 16, 17).

Wykorzystując liniowość modelu H-N oceniono wpływ efektu barogradientowego na pola przepływów. Wykazano, że stacjonarne, niejednorodne w przestrzeni pole ciśnienia atmosferycznego oddziałuje jedynie na pole poziomu morza, kształtując odpowiednio topografię powierzchni swobodnej morza. Przeprowadzone obliczenia przepływów barogradientowych stanowiły test dla oceny poprawności stosowanej metody badania przepływów ustalonych za pomocą nieustalonego układu równań wydatków masowych i poziomu morza.

Morze Bałtyckie charakteryzuje się znacznym dopływem wód rzecznych [5, 25]. Liniowość modelu H-N umożliwia ocenę cyrkulacji wód wywołanej wlewem rzek. Wykonano obliczenia wydatków masowych, poziomu morza i prądów na powierzchni Bałtyku dla tego typu cyrkulacji (rys. 23 — 25). Rezultaty obliczeń wskazują, że rola rzek w kształtowaniu poziomych ruchów wód ma charakter lokalny i ogranicza się do rejonów ujściowych oraz w pobliżu Cieśnin Duńskich, gdzie następuje odprowadzenie wód do Morza Północnego. Rozważając przepływy wiatrowe, zwłaszcza dla silnych wiatrów, można pominąć efekty związane z dopływem wód rzecznych.

Prezentowany w pracy model H-N może być z powodzeniem zastosowany do oceny przepływów wiatrowych w akwenach morskich o średniej głębokości.

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