Functional formalism for equations of Oberbeck-Boussinesq type of the developed thermohaline turbulence

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Abstract

The paper presents statistical description of weak turbulent convection in a linearly stratified, binary fluid. Problem of the thermohaline turbulent convection is formulated taking advantage of mutual characteristic functional of velocity, temperature and salinity fields. For two types of functionals, *viz.* STCF and SCF, some differential equations are presented, with functional derivatives. The equations comprise – as a specific case – equations presented by other authors [1, 23, 24].

1. Introduction

For various problems of ocean dynamics, sea water can be treated as a binary mixture of pure water and salt [10, 11]. Studies on the motions of such media must take into account instability of processes taking place in fluids (diffusion, thermal conductivity, and internal friction), and constitute subject of the thermodynamics of irreversible processes (comprehensive review of these problems has been given by Zubarev [27]). Thermodynamics of ocean waters was also discussed by Kamenkovich [10]; see also Landau and Lifszyc [15].

When temperature and salinity of sea waters in the gravitation fields are different in various points of the medium, hydrostatic equilibrium cannot occur in all spatial distributions of these parameters. In order to achieve the equilibrium it is necessary to have a constant and vertically directed gradient of temperature in the whole volume of the fluid, while salinity gradient must also be vertical and related to the temperature, *i.e.*:

$$\nabla T = -A_0 \vec{k}, \quad \nabla s = -B_0 \vec{k}, \quad \text{at } B_0 + \lambda A_0 = -\frac{\gamma}{\mu_s} \vec{g},$$

where:

A_0, B_0 – constants,

$\vec{k}$ – unit vector directed upwards,
λ – diffusion coefficient,
μ_s – chemical potential of salts,
$\gamma = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial s}\right)_{T,p}$
$\vec{g}$ – acceleration of gravity [26].

Lack of mechanical equilibrium leads to internal flows which transfer heat and salts, *i.e.* to complex phenomena of thermohaline convection. Mathematical description of these phenomena is based on the following set of equations [25, 26]:

$$\frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} = -\frac{\partial \pi}{\partial x_\alpha} + g_\alpha(\gamma_0 s - \alpha_0 \theta) + \nu \frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta} + f_\alpha(x, t),$$

$$\frac{\partial \theta}{\partial t} + u_\beta \frac{\partial \theta}{\partial x_\beta} = \zeta \frac{\partial^2 \theta}{\partial x_\beta \partial x_\beta} + \zeta \frac{\partial^2 s}{\partial x_\beta \partial x_\beta} + Q'_T(x, t), \tag{1}$$

$$\frac{\partial s}{\partial t} + u_\beta \frac{\partial s}{\partial x_\beta} = \mathfrak{z} \frac{\partial^2 \theta}{\partial x_\beta \partial x_\beta} + D \frac{\partial^2 s}{\partial x_\beta \partial x_\beta} + Q'_s(x, t),$$

$$\frac{\partial u_\beta}{\partial x_\beta} = 0, \text{ where } \alpha, \beta = 1, 2, 3.$$

In the equations (1) the following notations were used: $x = (x_1, x_2, x_3) \in \mathscr{D}$ where $\mathscr{D}$ is a domain in the three-dimensional Euclidean space R^3 with boundary $\partial \mathscr{D}$ (specifically $\mathscr{D}$ can be unbounded, *viz.* $\mathscr{D} = R^3$) $t \in [t_0, \infty)$; $u_\alpha(x, t)$ – field of fluid velocity; $\pi(x, t) = (p(x, t) - p_0)/\rho_0$, $\theta(x, t) = T(x, t) - T_0$, $s(x, t) = S(x, t) - s_0$ – where p, T, s are pressure, temperature and water salinity, respectively; "zero" index points to values of these parameters during equilibrium, at $\pi/p_0 \ll 1$, $\theta/T_0 \ll 1$, $S/s_0 \ll 1$. Furthermore, $f_\alpha(x, t)$, $Q'_T(x, t)$, $Q'_s(x, t)$ are assigned field of external irrotational forces, heat sources and salt sources, respectively; $\xi = \chi + N\lambda^2 D$, where χ – coefficient of thermal conductivity, $N = (T\mu_s/c_p)_0$; D – coefficient of salt diffusion; $\zeta = N\lambda D$; $\zeta = \lambda D$ (see [25]). Equations (1) have been obtained adopting several physical simplifications [10, 27]. Notwithstanding this, their mathematical structure is extremely complex. Note that assuming $\lambda = 0$ in the first and second equation of the set [1], *i.e.* taking no account of thermodiffusion, we will obtain the set of Oberbeck-Boussinesq equations, commonly used for the description of convection in a binary fluid [9].

It seems that analysis of the set of equations (1) with classical mathematical methods is essentially aimless, although in specific cases significant results were obtained using this set, most of all as regards the problems of transformation, *viz.* of the conditions in which laminar flows become turbulent flows [9]. Developed thermohaline convection necessitates statistical description; this being confirmed by direct observations of ocean dynamics [21]. In the essence, motions taking place in sea waters represent turbulence, though their nature is exceptionally complex due to the effect of earth rotation, stratification, existence of several zones of energy supply (differing as regards their scale). Nevertheless, it should be underlined that a general concept of turbulent transfer of the momentum, energy and mass, based on fundamentals of theoretical physics, has not been worked out as yet [8]. Theory of the turbulence is still under discussion; a comprehensive review of recent achievements in this field has been presented by Monin and Yaglom [19, 20], Leslie [16], and Monin and Ozmidov [21]. The present paper describes developed thermohaline turbulence taking advantage of reciprocal characteristic functional of the velocity, temperature and salinity fields. Possibility of statistical theory of turbulence, based on a characteristic functional of the velocity field, has been suggested for the first time by Kolmogorov [19], and taken advantage of by Hopf [4, 5]. Papers of the latter author constituted a starting point for many general works on functional formalism; their review is given in a monograph by Monin and Yaglom [20].

Other important works on these problems were published by Hosokawa [6, 7]. Problems of turbulent convection were also dealt with by Szafirski [24] who worked out equations for characteristic functional of velocity and temperature fields for a thermally stratified medium. Similar formalism was used by Ahmadi [1] for incompressible, homogeneous fluid. Moiseev et al. [17] presented an equation for compressible fluids, with specific selection of the state equation. More general equations than those given by Szafirski [24] were worked out by Sadontov [23]. However, in most cases [1, 17, 23] equations are given only for spatial characteristic functional. Furthermore, these authors were not as mathematically accurate as Szafirski [24], who also worked out equation for a space-time characteristic functional Equations presented in this paper are much more general than Szafirski's equations, so that they embrace also the latter ones as a specific case. Furthermore, with proper simplifications, these equations can easily be used to obtain the results presented by Ahmadi [1], and – if the relationship $T_0 = T_0(z)$ is taken for the set (1) of equations – – it is also possible to obtain Sadontov's results [23].

2. Formulation of the problem of turbulent thermohaline convection

(*i*) Let the field $[\vec{u}(x, t), \theta(x, t), s(x, t)]$ be a vector random field. Let's assume that $\bigwedge_{n \in N}$, and that for all points $M_1 = (x_1, t_1), \ldots, M_n = (x_n, t_n)$ such that $(x_i, t_i) \in \mathscr{D} \, x[t_0, \infty)$, $i = 1, \ldots, n$ there exists a probability density $p(M_1, \ldots, M_n)$ of the random vectors $[\vec{u}(M_1), \theta(M_1), s(M_1)], \ldots, [\vec{u}(M_n), \theta(M_n), s(M_n)]$. We can say that random vector field $[\vec{u}, \theta, s]$ is known if probability density $p(M_1, \ldots, M_n)$ is known for $\bigwedge_{n \in N}$ and all points $M_1, \ldots, M_n$ in $\mathscr{D} \, x[t_0, \infty)$. Let's assume that vector $[\vec{u}(x, t), \theta(x, t), s(x, t)$ satisfies the set of equations (1). Let's further assume that at the moment $t = t_0$ vector field $[\vec{u}(x, t_0), \theta(x, t_0), s(x, t_0)]$ such that $\partial u_\alpha / \partial x_\alpha = 0$ is given, *i.e.* we assume that for $\bigwedge_{n \in N}$ and for any points $\bar{M}_1 = (x_1, t_0), \ldots, \bar{M}_n = (x_n, t_0)$ initial functions $p(\bar{M}_1, \ldots, \bar{M}_n)$ are known. Then the problem of turbulent thermohaline convection should be formulated as follows: define time development of the vector random field $[\vec{u}(x, t), \theta(x, t), s(x, t)]$ *viz.* find the evolution in time of the function $p(M_1, \dots, M_n)$ for $\bigwedge_{n \in N}$ and for all points $M_1, \dots, M_n \in \mathscr{D} x [t_0; \infty)$.

Further on we shall see that when the problem is posed as above, it is possible to formulate it as a problem of probability distribution for proper hydrodynamic fields, taking advantage of characteristic functionals of these fields, which unequivocally define the above distributions over the phase space of turbulent flow.

(*ii*) Let $\Omega = \{\omega\}$ be the phase space of turbulent flow, *i.e.* a set the elements of which are represented by the vector fields $[\vec{u}(x, t), \theta(x, t), s(x, t)]$ satisfying the set of equations (1) and specified boundary conditions. We shall assume that at defined boundary conditions on the borders of the flow, the set of equations (1) uniquely defines the fields $[\vec{u}(x, t), \theta(x, t), s(x, t)]$, $t > t_0$, at given initial field $[\vec{u}(x, t_0), \theta(x, t_0), s(x, t_0)]$ (see [4, 5, 14, 18)]. Now we should precise (see [24]) these conditions. Let's consider a fixed vector field $[\vec{a}(x, t), b(x, t), c(x, t)]$ defined on the boundary $\partial\mathscr{D} x[t_0; \infty)$ of $\mathscr{D}x[t_0; \infty)$. We assume that the following boundary conditions are satisfied:

$$[\vec{u}(x, t), \theta(x, t), s(x, t)] = [\vec{a}(x, t), b(x, t), c(x, t)]$$

$$\text{on } \partial D \times [t_0; \infty). \tag{2.1}$$

When $\mathscr{D} = R^3$, we assume for Ω the following boundary condition at infinity:

$$\lim_{|x| \to \infty} [\vec{u}(x, t), \theta(x, t), s(x, t)] = \vec{p} \tag{2.2}$$

for all $[\vec{u}(x, t), \theta(x, t), s(x, t)] \in \Omega$, where $\vec{p}$ is a given constant vector, independent of all considered vector fields.

Let's assume that it is possible to introduce probability measure $P(\omega)$ on Ω *i.e.* a measure given on σ – algebra of Borel sets Ω, such that $P(A) \geqslant$, $A \subset \Omega$ countable additive and normalized $P(\Omega) = 1$. Let's also denote as Ω_0 the space in which the elements are represented by initial vector fields $[\vec{u}(x, t_0), \theta(x, t_0), s(x, t_0)]$ and assume that a measure $P_0(\omega_0)$ is given over this space, which defines probability of $[\vec{u}(x, t_0), \theta(x, t_0), s(x, t_0)]$ belonging to the Borel subset $\omega_0 \subset \Omega_0$. Problem of the turbulent thermohaline convection should be now formulated as follows: find the measure $P(\omega)$ concentrated on the set $\{[\vec{u}(x, t), \theta(x, t), s(x, t)]\}$ $(t \in [t_0; \infty))$ of the solutions to the system of equations (1), such that its contraction at $t = t_0$ would equal to measure P_0, *i.e.* $P([\vec{u}, \theta, s]; [\vec{u}(\cdot, t_0), \theta(\cdot, t_0), s(\cdot, t_0)] \in \omega_0) = P_0(\omega_0)$.

Measure P determined probability distribution on the set of solutions, which corresponds to the initial distribution and can be defined as a space – time statistical solution to the set of equations (1).

However, application of such a measure to an infinitively dimensional function space Ω is not simple (see[2]). In the infinitively dimensional case it is not possible to distinguish the distribution class having probability density, *viz.* there is no invariant measure under translations and rotations and would be easy to calculate in case of affine transformations. In particular, requirement of countable additivity of this measure is equivalent to the requirement of commutability of the integration of ω function against this measure, *viz.* of the operator of expected value of the functionals of $[\vec{u}(x,t), \theta(x,t), s(x,t)]$ with operations of the limiting passage, in this of differentiation and integration of these functionals against the parameter. In order to avoid consideration of quite inconvenient functions of the sets $P(S)$, $S \subset \Omega$, instead of $P(\omega)$ we shall use its continual transformation of Fourier, *viz.* a space-time characteristic functional (STCF) [18, 24].

$$\varphi[\vec{y}, p, q] = \langle \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\rangle = \int \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\}\rangle\, dP, \tag{2.3}$$

where

$$\{\vec{y}, p, q; \vec{u}, \theta, s\} = \{\vec{y}(x,t), p(x,t), q(x,t); \vec{u}(x,t), \theta(x,t), s(x,t)\} =$$
$$= \int [y_j(x,t)u_j(x,t) + p(x,t)\theta(x,t) + q(x,t)s(x,t)]\, dxdt \tag{2.4}$$

$\mathscr{D}x[t_0; \infty)$

at $[\vec{y}(x,t), p(x,t), q(x,t)] \in H$, where H is the space of continuous vector fields on $\mathscr{D}x[t_0; \infty)$ with compact support in $\mathscr{D}x[t_0; \infty)$. Values of the functional $\varphi[\vec{y}, p, q]$ in the "improper points" of the H space.

$$\tilde{y}_B(x,t) = \sum_{j=1}^{n} R_j \delta_{\alpha_j\beta}\, \delta(x-x_j)\delta(t-t_j),$$

$$\tilde{p}(x,t) = \sum_{j=1}^{n} R_j \delta_{\alpha_j 4}\, \delta(x-x_j)\delta(t-t_j),$$

$$\tilde{q}(x,t) = \sum_{j=1}^{n} R_j \delta_{\alpha_j 5}\, \delta(x-x_j)\delta(t-t_j), \tag{2.5}$$

where $\delta_{\alpha_j\beta}$ is the Kronecker's delta; $\delta(\cdot)$ – Dirac distribution $R_1, \ldots, R_n \in R^1$; $d_j = 1, \ldots, 5$; $\beta = 1, 2, 3$ covers with characteristic functions of probability distributions for values $[\vec{u}(x_j, t_j), \theta(x_j, t_j), s(x_j, t_j)]$ of the field $[\vec{u}(x,t), \theta(x,t), s(x,t)]$ over limited number of space-time points $(x_1, t_1), \ldots, (x_n, t_n) \in \mathscr{D}x[t_0; \infty)$. It is now easy to see [14, 24] that:

$$\varphi[\tilde{\vec{y}}, \tilde{p}, \tilde{q}] = \langle \exp[i \sum_{\substack{j=1 \\ \alpha_j \leq 3}}^{n} R_j u_{\alpha_j}(x_j, t_j) + \sum_{j=1}^{n} R_j \delta_{\alpha_j 4}\, \theta(x_j, t_j) +$$
$$+ \sum_{j=1}^{n} R_j \delta_{\alpha_j 5}\, s(x_j, t_j)]\rangle, \tag{2.6}$$

viz. $\varphi[\tilde{\vec{y}}, \tilde{p}, \tilde{q}]$ is a characteristic function of n-dimensional probability distribution of random variable $[\eta_{\alpha_1}(x_1, t_1), \ldots \eta_{\alpha_n}(x_n, t_n)]$, where $\eta_{\alpha_m}(x_m, t_m) = u_{\alpha_m}(x_m, t_m)$ for $\alpha_m \leq 3$, $\eta_{\alpha_m}(x_m, t_m) = \theta(x_m, t_m)$ for $\alpha_m = 4$ and $\eta_\alpha(x_m, t_m) = s(x_m, t_m)$ for $\alpha_m = 5$. Introducing the Fourier transformation of characteristic function (2.6), we can obtain the function of probability density $p(\eta_1, \ldots, \eta_n)$ defined as:

$$p(\eta_1, \ldots, \eta_n) = (2\pi)^{-n} \int \exp(-i \sum_{j=1}^{n} R_j \eta_j)\, \varphi[\tilde{\vec{y}}, \tilde{p}, \tilde{q}]\, dR_1, \ldots, dR_n. \tag{2.7}$$

Thus, all finite dimensional probability distributions for the field $[\vec{u}(x,t), \theta(x,t), s(x,t)]$ can be uniquely determined basing on the functional $\varphi[\vec{y}, p, q]$.

(*iii*). Let's introduce a definition of the differentiation the functional $\varphi[\vec{y}, p, q]$. Since φ is the functional dependent of five functions y_1, y_2, y_3, p, q, then introducing the notation y_α, $\alpha=1, \ldots, 5$; $y_4=p$, $y_5=q$ we can formulate the following definition: Definition 1. Functional $\varphi[..y_\alpha..]$ is differentiable against function $y_\alpha=y_\alpha(x, t)$ if there exists a function $A_\alpha[y_\beta; x, t]$ such that:

$$\varphi[\ldots y_\alpha+\delta y_\alpha\ldots]-\varphi[\ldots y_\alpha\ldots]=\int_{\mathscr{D}}\int_{t_0}^{\infty} A_\alpha[y_\alpha; x, t]\,\delta y_\alpha(x, t)\,dxdt+$$

$$+\bar{o}\Big[\int_{\mathscr{D}}\int_{t_0}^{\infty}|\delta y_\alpha(x, t)|\,dxdt\Big]. \tag{2.8}$$

Functional $A_\alpha[y_\beta; x, t]$ is called a partial functional variational derivative of φ with respect to y_α at the point (x, t). It is readily visible that expression (2.8) can also be presented in an equivalent form [24].

$$\frac{\delta\varphi[\ldots y_\alpha\ldots]}{\delta y_\alpha(x, t)}=\lim_{\substack{sup\,|\delta y_\alpha(x,t)|\to 0\\ x\in D\\ t\geqslant t_0}}\frac{\varphi[\ldots y_\alpha(x, t)+\delta y_\alpha(x, t)\ldots]-\varphi[\ldots y_\alpha(x, t)\ldots]}{\int_{\mathscr{D}}\int_{t_0}^{\infty}\delta y_\alpha(x, t)\,dxdt}$$

in which:

$$A_\alpha[y_\alpha; x, t]=\frac{\delta\varphi[\ldots y_\alpha\ldots]}{\delta y_\alpha(x, t)}\,. \tag{2.9}$$

Taking advantage of the definition 1 we can see that:

$$\frac{\delta y_\alpha(x', t')}{\partial y_\beta(x, t)}=\delta_{\alpha\beta}\,\delta(x-x')\,\delta(t-t'), \tag{2.10}$$

where $\delta_{\alpha\beta}$ is the Kronecker's delta.

3. Equations for space-time characteristic functional

Now let's derive functional differential equations for $\varphi[\vec{y}, p, q]$. To simplify the problem we shall omit functional arguments in the expressions for functional derivatives (2.9).

Note that

$$\frac{\delta\varphi}{\delta y_\alpha}=\langle iu_\alpha\exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\rangle, \tag{3.1}$$

$$\frac{\delta\varphi}{\delta p}=\langle i\theta\exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\rangle, \tag{3.2}$$

$$\frac{\delta\varphi}{\delta q}=\langle is\exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\rangle, \tag{3.3}$$

$$\frac{\delta^2\varphi}{\delta y_\alpha\,\delta y_\beta}=\langle -u_\alpha u_\beta\exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\rangle, \tag{3.4}$$

$$\frac{\delta^2 \varphi}{\delta y_\beta \, \delta p} = \langle -u_\beta \theta \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\rangle, \tag{3.5}$$

$$\frac{\delta^2 \varphi}{\delta y_\beta \, \delta q} = \langle -u_\beta s \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\rangle. \tag{3.6}$$

Dependencies (3.1) – (3.6) were obtained taking advantage of the definition 1, and expression (2.10).
Taking into account system of equations (1) we obtain:

$$\frac{\partial}{\partial t}\frac{\delta \varphi}{\delta y_\alpha} = \left\langle i \frac{\partial u_\alpha}{\partial t} \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle = \left\langle i\left[-\frac{\partial \pi}{\partial x_\alpha} - u_\beta \frac{\partial u_\alpha}{\partial x_\beta} + \right.\right.$$

$$\left.\left. + g_\alpha(\gamma_0 s - \alpha_0 \theta) + v \frac{\partial^2 u_\alpha}{\partial x_\beta \, \partial x_\beta} + f_\alpha\right] \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle,$$

$$\frac{\partial}{\partial t}\frac{\delta \varphi}{\delta p} = \left\langle i \frac{\partial \theta}{\partial t} \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle = \left\langle i\left[-u_\beta \frac{\partial \theta}{\partial x_\beta} + \xi \frac{\partial^2 \theta}{\partial x_\beta \partial x_\beta} + \right.\right.$$

$$\left.\left. + \zeta \frac{\partial^2 s}{\partial x_\beta \, \partial x_\beta} + Q_r'\right] \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle,$$

$$\frac{\partial}{\partial t}\frac{\delta \varphi}{\delta q} = \left\langle i \frac{\partial s}{\partial t} \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle = \left\langle i\left[-u_\beta \frac{\partial s}{\partial x_\beta} + \zeta \frac{\partial^2 \theta}{\partial x_\beta \partial x_\beta} + \right.\right.$$

$$\left.\left. + D \frac{\partial^2 s}{\partial x_\beta \, \partial x_\beta} + Q_s'\right] \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle.$$

But:

$$\frac{\partial^2}{\partial x_\beta \, \partial x_\beta}\frac{\delta \varphi}{\delta y_\alpha} = \left\langle i \frac{\partial^2 u_\alpha}{\partial x_\beta \, \partial x_\beta} \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle, \tag{3.7}$$

$$\frac{\partial}{\partial x_\beta}\frac{\delta^2 \varphi}{\delta y_\alpha \delta y_\beta} = \left\langle -u_\beta \frac{\partial u_\alpha}{\partial x_\beta} \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle, \tag{3.8}$$

$$\frac{\partial^2}{\partial x_\beta \, \partial x_\beta}\frac{\delta \varphi}{\delta p} = \left\langle i \frac{\partial^2 \theta}{\partial x_\beta \, \partial x_\beta} \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle, \tag{3.9}$$

$$\frac{\partial}{\partial x_\beta}\frac{\delta^2 \varphi}{\delta y_\beta \delta p} = \left\langle -u_\beta \frac{\partial \theta}{\partial x_\beta} \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle, \tag{3.10}$$

$$\frac{\partial^2}{\partial x_\beta \, \partial x_\beta}\frac{\delta \varphi}{\delta q} = \left\langle i \frac{\partial^2 s}{\partial x_\beta \, \partial x_\beta} \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle, \tag{3.11}$$

$$\frac{\partial}{\partial x_\beta}\frac{\delta^2 \varphi}{\delta y_\beta \delta q} = \left\langle -u_\beta \frac{\partial s}{\partial x_\beta} \exp(i\{\vec{y}, p, q; \vec{u}, \theta, s\})\right\rangle. \tag{3.12}$$

Now, taking into consideration dependencies (3.7) - (3.12) we have:

$$\frac{\partial}{\partial t}\frac{\delta\varphi}{\delta y_\alpha}=i\frac{\partial}{\partial x_\beta}\frac{\delta^2\varphi}{\delta y_\alpha\delta y_\beta}+\nu\frac{\partial^2}{\partial x_\beta\partial x_\beta}\frac{\delta\varphi}{\delta y_\alpha}+\gamma_0 g_\alpha\frac{\delta\varphi}{\delta q}-\alpha_0 g_\alpha\frac{\delta\varphi}{\delta p}+F_\alpha-\frac{\partial\Pi}{\partial x_\alpha} \tag{3.13}$$

where

$$F_\alpha=\langle if_\alpha\exp(i\{\vec{y},p,q;\vec{u},\theta,s\})\rangle,$$
$$\Pi=\langle i\pi\exp(i\{\vec{y},p,q;\vec{u},\theta,s\})\rangle; \tag{3.14}$$

and

$$\frac{\partial}{\partial t}\frac{\delta\varphi}{\delta p}=i\frac{\partial}{\partial x_\beta}\frac{\delta^2\varphi}{\delta y_\alpha\delta p}+\xi\frac{\partial^2}{\partial x_\beta\partial x_\beta}\frac{\delta\varphi}{\delta p}+\zeta\frac{\partial^2}{\partial x_\beta\partial x_\beta}\frac{\delta\varphi}{\delta q}+Q_T, \tag{3.15}$$

where

$$Q_T=\langle iQ'_T\exp(i\{\vec{y},p,q;\vec{u},\theta,s\})\rangle; \tag{3.16}$$

and also

$$\frac{\partial}{\partial t}\frac{\delta\varphi}{\delta q}=i\frac{\partial}{\partial x_\beta}\frac{\delta^2\varphi}{\delta y_\beta\delta q}+\zeta\frac{\partial^2}{\partial x_\beta\partial x_\beta}\frac{\delta\varphi}{\delta p}+D\frac{\partial^2}{\partial x_\beta\partial x_\beta}\frac{\delta\varphi}{\delta q}+Q_s, \tag{3.17}$$

where

$$Q_s=\langle iQ'_s\exp(i\{\vec{y},p,q;\vec{u},\theta,s\})\rangle. \tag{3.18}$$

When (3.1) is taken into consideration the continuity equation $\partial u_\beta/\partial x_\beta=0$ gives:

$$\frac{\partial}{\partial x_\beta}\frac{\delta\varphi}{\delta y_\beta}=0. \tag{3.19}$$

It is now quite easy to show that the functional $\varphi[\vec{y},p,q]$ is invariant as regards transformation $\mathscr{L}$ in the space of function $\vec{y}$, *viz.* there exists such linear operator $\mathscr{L}$, $\mathscr{L}\vec{y}(x,t)=\tilde{\vec{y}}(x,t)$ that [20, 24]:

$$\varphi[\vec{y},p,q]=\varphi[\vec{y},p,q], \tag{3.20}$$

where $\vec{y}(x,t)$ is the solenoidal part of the vector $\vec{y}(x,t)$; $\partial\breve{y}_\alpha/\partial x_\alpha=0$ and $\breve{y}_n/_{\partial\tilde{E}_t}=0$ ($\tilde{E}_t\subset\mathscr{D}\wedge\bar{E}_t\subset\tilde{E}_t\wedge\bar{E}_t\subset\mathscr{D}$, where $\vec{y}(x,t)$ is a continuous vector on $\mathscr{D}x[t_0;\infty)$ which for fixed t – disappears outside $\tilde{E}_t$ such that $\bar{E}_t\subset\mathscr{D}$).

Property (3.20) is quite important as it makes possible to get rid of the term referring to pressure in the equation (3.13). In order to do so we must multiply (meaning make a scalar product) both sides of (3.13) by the function $\vec{\eta}(x,t)$, which satisfies condition $\partial\eta_\alpha/\partial x_\alpha=0$ and disappears sufficiently rapidly at $|x|\to\infty$ (especially $\vec{\breve{y}}(x,t)$ can be used as function $\vec{\eta}$).

In view of the fact that $\int_{\mathscr{D}} \eta_\alpha(x,t)\frac{\partial \Pi}{\partial x_\alpha}=0$. we obtain:

$$\int_{\mathscr{D}} \eta_\alpha(x,t)\left[\frac{\partial}{\partial t}\frac{\delta\varphi}{\delta y_\alpha}-i\frac{\partial}{\partial x_\beta}\frac{\delta^2\varphi}{\delta y_\alpha \delta y_\beta}-\nu\frac{\partial^2}{\partial x_\beta \partial x_\beta}\frac{\delta\varphi}{\delta y_\alpha}-\right.$$

$$\left.+\nu_0 g_\alpha\frac{\delta\varphi}{\delta q}+\alpha_0 g_\alpha\frac{\delta\varphi}{\delta p}-F_\alpha\right]dxdt=0. \tag{3.21}$$

Equations (3.15), (3.17) and (3.21) represent the sought equations for STCF $\varphi[\vec{y},p,q]$. Moreover, functional $\varphi[\vec{y},p,q]$ must satisfy the following conditions [14, 24]:

$$\varphi[\bar{0},0,0]=1\,;\qquad \varphi^*[\vec{y},p,q]=\varphi[-\vec{y},p,q]\,;\qquad |\varphi[\vec{y},p,q]|\leqslant 1. \tag{3.22}$$

Here * denotes the complex conjugate. The first two conditions result directly from (2.3) and P normalization. The third condition results from normalization and non--negativity of P. It constitutes the simplest condition from an infinite sequence of inequality, which must be satisfied by φ, and which results from positive definiteness $\varphi[\vec{y},p,q]$ [14].

Denoting by $P_0(A)$ the probability distribution defined on subsets of vector field spaces $[\vec{u}(x,t_0),\ \theta(x,t_0),\ s(x,t_0)]$ such that $\partial u_\beta/\partial x_\beta=0$, we can now formulate the problem of turbulent thermohaline convection as follows: find the solution to equations (3.15), (3.17), (3.21) which would satisfy condition (3.22) and the following initial condition:

$$\varphi[\vec{y}(x)\delta(t-t_0),\,p(x)\delta(t-t_0),\,q(x)\delta(t-t_0)]=\varphi_0[\vec{y}(x),\,p(x),\,q(x)], \tag{3.23}$$

where $\varphi_0[\vec{y},p,q]$ is the characteristic functional of the probability distribution $P_0(A)$.

4. Equation for space characteristic functional

Less comprehensive statistical description of the field $[\vec{u}(x,t),\ \theta(x,t),\ s(x,t)]$ can be obtained defining space characteristic functional SCF [20, 24]:

$$\varphi[\vec{y},p,q\,;\,t]=\int \exp(i\{\vec{y},p,q\,;\,\vec{u},\theta,s\,;\,t\})dP=$$
$$=\langle\exp(i\{\vec{y},p,q\,;\,\vec{u},\theta,s\,;\,t\})\rangle, \tag{4.1}$$

where

$$\{\vec{y},p,q\,;\,\vec{u},\theta,s\,;\,t\}=\int_D Cy_\alpha(x)u_\alpha(x,t)+p(x)\theta(x,t)+q(x)s(x,t)]dx, \tag{4.2}$$

and where $[\vec{y}(x),\ p(x),\ q(x)]$ belongs to space H_0 of vector continuous fields on $\mathscr{D}$, with a compact support in D.

Considerations presented in previous paragraphs are valid also for this case, obviously with proper modifications. They can be quite easily obtained (see [24]). Let's derive equation for SCF $\varphi[\vec{y},p,q,t]$. Note that $\varphi[\vec{y},p,q;t]$ represents now functional from $\vec{y},p,q$ and for fixed functions $\vec{y},p,q$ it constitutes a time function t.

Thus:

$$\frac{\partial\varphi}{\partial t}=\left\langle i\int_{\mathscr{D}}\left[y_\alpha(x)\frac{\partial u_\alpha}{\partial t}+p(x)\frac{\partial\theta}{\partial t}+q(x)\frac{\partial s}{\partial t}\right]\exp(i\{\vec{y},p,q;\vec{u},\theta,s;t\})\right\rangle.$$

Using the system of equations (1) we have:

$$\frac{\partial\varphi}{\partial t}=\left\langle i\int_{\mathscr{D}}\left\{y_\alpha\left[-u_\beta\frac{\partial u_\alpha}{\partial x_\beta}-\frac{\partial\pi}{\partial x_\alpha}+\nu\frac{\partial^2u_\alpha}{\partial x_\beta\,\partial x_\beta}+g_\alpha\gamma_0 s-g_\alpha\alpha_0\theta+f_\alpha\right]+\right.\right.$$

$$+p\left[-u_\beta\frac{\partial\theta}{\partial x_\beta}+\xi\frac{\partial^2\theta}{\partial x_\beta\,\partial x_\beta}+\zeta\frac{\partial^2 s}{\partial x_\beta\,\partial x_\beta}+Q'_T\right]+q\left[-u_\beta\frac{\partial s}{\partial x_\beta}+\zeta\frac{\partial^2\theta}{\partial x_\beta\partial x_\beta}+\right.$$

$$\left.\left.+D\frac{\partial^2 s}{\partial x_\beta\,\partial x_\beta}+Q'_s\right]dx\exp(i\{\vec{y},p,q;\vec{u},\theta,s;t\})\right\rangle.$$

Now we must express quantities at the right-hand side of the above dependence in terms of functional derivatives. In order to do so, advantage should be taken of equations analogous to (3.1)–(3.6), which can be obtained from the definition (4.1) and rules of functional differentiation. And thus:

$$\frac{\partial\varphi}{\partial t}=\int_{\mathscr{D}}\left\{y_\alpha(x)\left[i\frac{\partial}{\partial x_\beta}\frac{\delta^2\varphi}{\delta y_\alpha\delta y_\beta}+\nu\frac{\partial^2}{\partial x_\beta\,\partial x_\beta}\frac{\delta\varphi}{\delta y_\alpha}+g_\alpha\gamma_0\frac{\delta\varphi}{\delta q}-g_\alpha\alpha_0\frac{\delta\varphi}{\delta p}+F_\alpha\right]+\right.$$

$$+p(x)\left[i\frac{\partial}{\partial x_\beta}\frac{\delta^2\varphi}{\delta y_\beta\delta p}+\xi\frac{\partial^2}{\partial x_\beta\,\partial x_\beta}\frac{\delta\varphi}{\delta p}+\zeta\frac{\partial^2}{\partial x_\beta\,\partial x_\beta}\frac{\delta\varphi}{\delta q}+Q_T\right]+ \tag{4.3}$$

$$\left.+q(x)\left[i\frac{\partial}{\partial x_\beta}\frac{\delta^2\varphi}{\delta y_\beta\delta q}+\zeta\frac{\partial^2}{\partial x_\beta\,\partial x_\beta}\frac{\delta\varphi}{\delta p}+D\frac{\partial^2}{\partial x_\beta\,\partial x_\beta}\frac{\delta\varphi}{\delta q}+Q_s\right]\right\}dx.$$

Equation (4.3) is the sought equation for SCF $\varphi[\vec{y},p,q;t]$.

In the above (4.3) solenoidality of the field $\vec{y}(x)$ has been taken into account. This way it has been possible to eliminate the term expressing pressure. The following denotations have also been introduced:

$$F_\alpha=\langle if_\alpha\exp(i\{\vec{y},p,q;\vec{u},\theta,s;t\})\rangle,$$

$$Q_T=\langle iQ'_T\exp(i\{\vec{y},p,q;\vec{u},\theta,s;t\})\rangle,$$

$$Q_s=\langle iQ'_s\exp(i\{\vec{y},p,q;\vec{u},\theta,s;t\})\rangle.$$

Equation (4.3) possesses first order, in time, so that it can be used to determine functional $\varphi[\vec{y}(x), p(x), q(x); t]$, at initial condition:

$$\varphi[\vec{y}(x),p(x),q(x);t_0]=\varphi_0[\vec{y}(x),p(x),q(x)]. \tag{4.4}$$

Functional $\varphi_0[\vec{y}(x), p(x), q(x),]$ must satisfy restrictions resulting from the condition of incompressibility $\partial/\partial x_\beta(\delta/\delta y_\beta\varphi_0)=0$. In this case solution to equation (4.3) $\varphi[\vec{y}(x), p(x), q(x), t]$ automatically satisfies this condition, at any $t>t_0$ [20].

5. Final remarks

The presented functional formalism, applied to the problem of weak turbulent convection in a linearly stratified binary fluid, allows for a most constructive and consistent approach to the exceptionally complex physical phenomena taking place in case of such convection.

This results from the fact that equations for the characteristic functional are equivalent to an infinite set of stochastic differential equations, which would be rather difficult to analyse without the use of functional calculus.

Special feature of the adopted approach consists of the fact that linear equations are obtained for the φ functional. Due to insufficient knowledge as regards the theory of differential equations with variational derivatives, it is not possible to obtain practical results from these equations as yet. Nevertheless, possibility of comprehensive description of the field $[\vec{u}(x, t), \theta(x, t), s(x, t)]$, treated as a vector stochastic field, with one value only, *viz.* mutual characteristic functional, seems to be of uttermost importance.

Moreover, such description allows for many interesting observations; most of all, formulation of several problems of statistical hydromechanics is in many aspects analogous to the formulation of problems in quantum field theory [3, 20, 22] and statistical mechanics [22]. Functional formalism also constitutes effective approach to quantum optics and theory of dynamic systems [12, 13].

Some resulting analogies with ocean turbulence shall be presented in next papers.

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