R O Z P R A W Y

PAPERS

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THE ESTIMATION AND ACCURACY OF CHARACTERISTICS OF RANDOM PROCESSES

The theory of random processes finds increasing application in oceanography and hydrology, therefore it is expedient to realize the errors inherent in particular data processing techniques. A knowledge of the errors enables one to choose the most appropriate procedures to estimate the characteristics of random processes and the confidence intervals of the estimates.

It is assumed throughout this paper that one deals with an analogue record or digital series of an ergodic random process X(t). For the sake of simplicity the formulae are presented only for the analogue record, which is subject to digital quantization.

Let us start from a closer examination of the formulae for the estimate of the expected (or mean) value of a random process, which will make it easier to observe the other random characteristics. This estimate, for averaging over the time T, is another random parameter

$$m_x^* = \frac{1}{T} \int_{t_0}^{t_0+T} X(t)\, dt \tag{1}$$

By evaluating different statistical moments on the basis of the probability calculus theorems one is able to determine the estimates of correlation functions, spectral densities, etc., that is to show how these correlation functions etc. computed from a single random series represent the whole statistical population. It can thus be proved that the expected value M of the mean value m_x^* of a random process X(t) is a true mean value m_x

$$M[m_x^*] = m_x \tag{2}$$

The variance of the mean value m_x^* in the realm of the whole population of m_x^* (i.e. the measure of the approximation of m_x^* to the true value m_x) is

$$\sigma_m^2 = \frac{1}{T^2} \int_{t_o}^{t_o+T} \int_{t_o}^{t_o+T} K_{xx}(t_2 - t_1)\, dt_1\, dt_2 = \frac{2}{T} - \int_0^T \left(1 - \frac{\tau}{T}\right) K_{xx}(\tau)\, d\tau \tag{3}$$

$$\text{for} \lim_{T \to \infty} \sigma_m^2 = 0 \tag{4}$$

which gives

$$\sigma_m = \sigma_x \sqrt{\frac{2}{\alpha T}\left[1 - \frac{1}{\alpha T}(1 - e^{-\alpha T})\right]} \tag{5.1.}$$

or

$$\sigma_m \approx \sigma_x \sqrt{\frac{2}{\alpha T}} \quad \text{for } \alpha T \gg 1 \tag{5.2.}$$

in the case of the correlation function

$$K_{xx}(\tau) = \sigma_x^2 e^{-\alpha |T|} \tag{5.3.}$$

and

$$\sigma_m \approx \sigma_x \sqrt{\frac{2}{\alpha T\left[1 + \left(\frac{\omega_o}{\alpha}\right)^2\right]}} \quad \text{for } \alpha T \gg 1 \tag{6.1.}$$

if the correlation function is

$$K_{xx}(\tau) = \sigma_x^2 e^{-\alpha |T|} \cos \omega_o \tau \tag{6.2.}$$

By analogy, one can analyse the propriety of representation of the correlation functions of a random process. For a given mean value m_x one obtains the following compatible estimate of the correlation function:

$$M\,[K_{xx}^*(\tau)] = K_{xx}(\tau) \tag{7}$$

and the variance of the correlation function

$$\sigma_k^2(\tau) = \frac{2}{(T-\tau)^2} \int_0^{T-\tau} (T - \tau - \tau_1)\ [K_{xx}^2(\tau_1) + K_{xx}(\tau_1 + \tau)\cdot K_{xx}(\tau_1 - \tau)]\, d\tau_1 \tag{8}$$

Upon consideration of the dominant interval of the correlation function $0 < \tau < \tau_{max}$ and long time period T one obtains

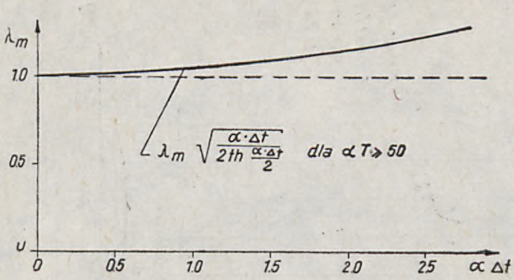

Fig. 1. The variation of $\lambda = \frac{\delta_m \text{ (after Eq. 13)}}{\delta_m \text{ (after Eq. 9)}}$ with the ratio $\frac{\alpha T}{N}$

Rys. 1. Zależność $\lambda m = \frac{\sigma_m \text{ (wg 13)}}{\sigma_m \text{ (wg 9)}}$ od stosunku $\frac{\alpha T}{N}$ $\lambda_m = \sqrt{\frac{\alpha T}{2N} \operatorname{cth} \frac{\alpha T}{2N}}$

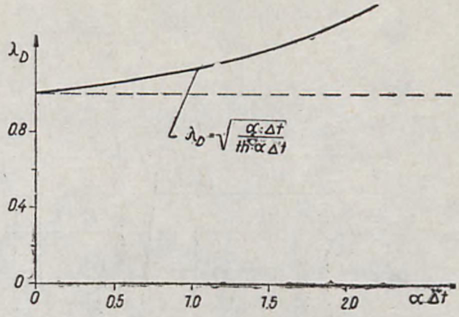

Fig. 2. The variation of $\lambda_D = \frac{\delta_D \text{ (after Eq. 20')}}{\delta_D \text{ (after Eq. 20)}}$ with the ratio $\frac{\alpha T}{N}$

Rys. 2. Zależność $\lambda_D = \frac{\sigma_D \text{ (wg)}}{\sigma_D \text{ (wg)}}$ od stosunku $\frac{\alpha T}{N}$ $\lambda_D = \sqrt{\frac{\alpha T}{N} \operatorname{cth} \frac{\alpha T}{N}}$

$$\sigma_k^2 (\tau) \approx \frac{1}{T - \tau} \int_{-\infty}^{+\infty} [K_{xx}^2 (\tau_1) + K_{xx} (\tau_1 + \tau) K_{xx} (\tau_1 - \tau)] \, d\tau_1 \tag{9}$$

For the correlation function of the type (5.3) one obtains from (9)

$$\sigma_K^2 (\tau) \approx \frac{D_{xx}^2}{\alpha (T - \tau)} [1 + (1 + 2 \alpha \tau) e^{-2\alpha\tau}] \tag{10}$$

Characteristic for many oceanographical and hydrological processes is the correlation function (similar to 6.2.)

$$K_{xx} (\tau) = \sigma_x^2 \, e^{-\alpha|\tau|} \cos \beta \tau \cdot \cos \omega_o \tau \tag{11}$$

The statistical scatter of this function is measured through a complex product σ_k^2 of $\frac{D_{xx}}{4 (T - \tau)}$ and a combination of an exponential function, trigonometric sine and cosine functions, and power polynomials of α, β, ω_o, and τ.

From the analysis of this formula it follows that σ_k^2

i — decreases with increasing τ

ii — is highest at the extrema of $K_{xx} (\tau)$

iii — is lowest at the zero crossings of $K_{xx} (\tau)$

iv — reaches the highest possible value for $K_{xx}(0)$

v — is about half of its maximum value for

$$\tau = \frac{2}{\alpha} \text{ (still with } T \gg \tau)$$

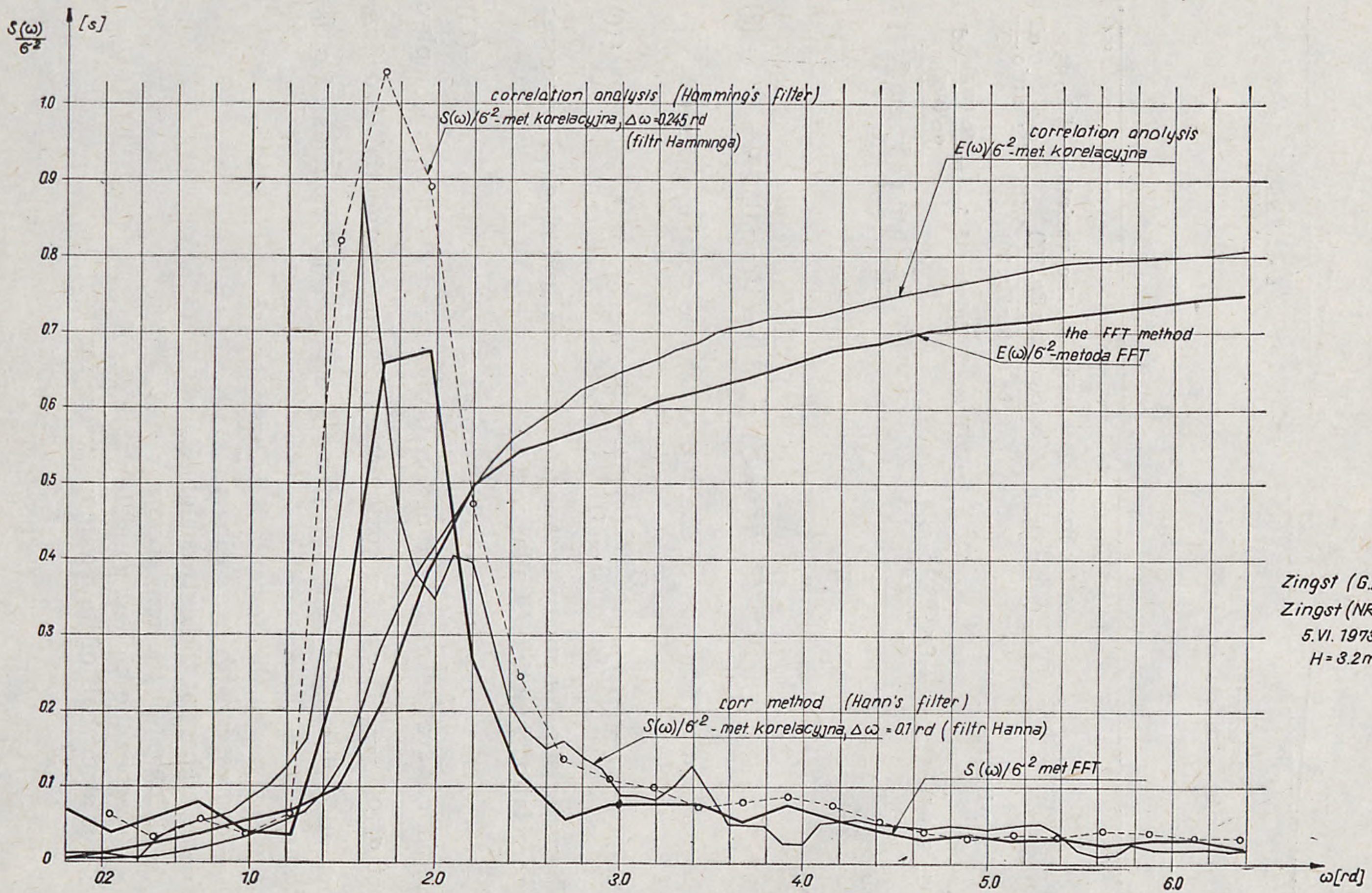

Fig. 3. Comparison of the spectra for the correlation method and FFT

Rys. 3. Porównanie widm dla metody korelacyjnej i FFT

Although the deviation σ^2_{kx} decreases with τ, the relative error

$$Q_{K^*(\tau)} = \left(\frac{\sigma^2_{K^*_{xx}}(\tau)}{K^*_{xx}(\tau)}\right)^{1/2} \tag{12}$$

increases with time and after the time τ_{cr} corresponding to

$$T = \frac{2\alpha^2 + \omega_o^2}{2\alpha(\alpha^2 + \omega_o^2)}\left[1 + \exp(2\alpha\tau_{cr})\right] + 2\tau_{cr} \tag{13}$$

the function $K^*_{xx}(\tau)$ does not carry any useful information.

If the mean value m_x is not known à priori, a long sample (long T) is required, but this stipulation is almost always satisfied for other reasons (e.g. with respect to spectral density).

The estimate $\widetilde{S}(j \cdot \Delta\omega)$, derived from the correlation function by the Wiener-Khinchine transformation, is not the most effective (i.e. with smallest variance in the general population of random $\widetilde{S}$) estimate of the true spectral density S (ω). In other words, the variability of the estimates of spectral densities does not change with the length of random sample and the length of the correlation function transformed. As a result, the function $\widetilde{S}(j \cdot \Delta\omega)$ must be smoothed out or is to be computed from the correlation function with an appropriate weighing function. It can be readily proved that both operations (smoothing and weighing) are equivalent (cf. Polish paper in Studia i Mat. Ocean. 12).

As transforms of random functions, the spectral densities are also random. The finite length of time series brings about the fact that the expected value of spectral density is not the true value from the realm of infinite series:

$$M\left[S^*_{xx}(\omega)\right] = \frac{1}{\pi}\int_0^\infty K_{xx}(\tau)\cos\omega\tau\, d\tau + \frac{1}{\pi}\int_{\tau_{max}}^\infty \left[K_{xxex}(\tau) - \right.$$

$$\left. - K_{xx}(\tau)\right]\cdot\cos\omega\tau\, d\tau \tag{14}$$

The second RHS term discribes the "noise of finiteness" due to the approximation of $S^*_{xx}(\omega)$ by integration in the finite interval $0 < \tau < \tau_{max}$. The function K_{xxex} describes the correlation values extrapolated beyond the interval $0 < \tau < \tau_{max}$.

By substituting any correlation functions in Fig. 4 one may determine the effect of cutting off $K(\tau)$ at $\tau = \tau_{max}$. An example for $K_{xx}(\tau)$ in the form (5.3) is shown in Fig. 5, whilst Fig. 6 illustrates the same effect for the function (11.1).

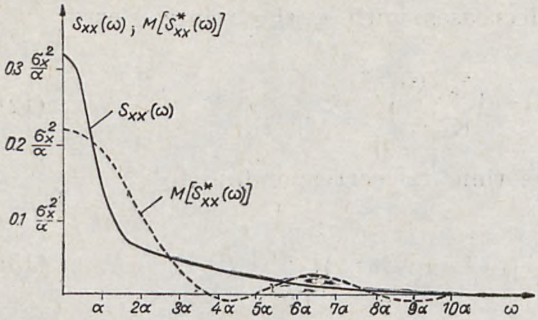

Fig. 4. The deviation of the spectral density $S^*_{xx}(\omega)$ from the true value $S_{xx}(\omega)$ due to the assumption $K^*_{xx}(\tau) = 0$ for $\tau > \tau_{max}$

Rys. 4. Odmienność realizacji gęstości widmowej $S^*_{xx}(\omega)$ od wartości prawdziwej $S_{xx}(\omega)$ w następstwie założenia $K^*_{xx}(\tau) = 0$ dla $\tau > \tau_{max}$

In order to best approximate the estimate of spectral density it is recommended that

the correlation function be extended to cross with the abscissa axis (of. Fig. 4),

the correlation length τ_{max} be greater than $\frac{3}{\alpha}$.

If the spectral density is computed by any method of numerical integration, it is well aproximated within the interval of frequencies lower than the Nyquist-Kotelnikov frequency

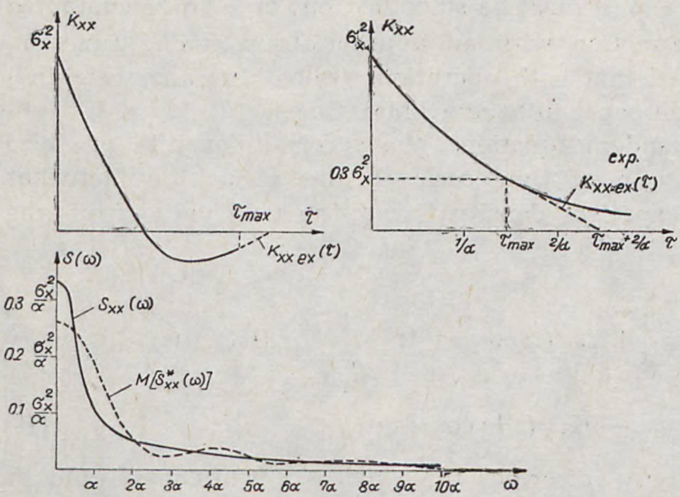

Fig. 5. The spectral density $S^*_{xx}(\omega)$ obtained from the correlation function extended to the abscissa axis

Rys. 5. Funkcja gęstości widmowej $S^*_{xx}(\omega)$ jako wynik przedłużania funkcji korelacji do przecięcia z osią odciętych

$$f_N = \frac{1}{2\,\Delta t} \tag{15.1.}$$

or

$$\Delta t < \frac{1}{2\,f_N} \tag{15.2.}$$

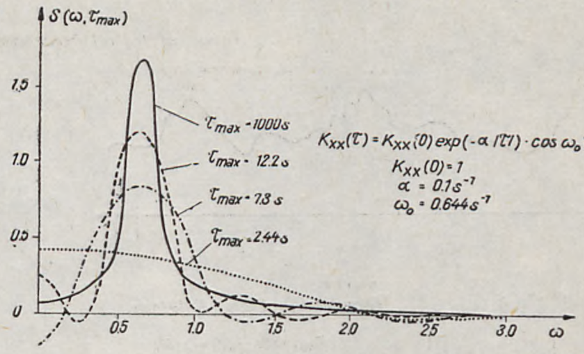

Fig. 6. The effect of time τ_{max} on the form of the spectral density $S^*_{xx}(\omega)$ for K_{xx} after Eq. 10

Rys. 6. Wpływ czasu τ_{max} na postać funkcji gęstości widmowej $S^*_{xx}(\omega)$ dla K_{xx} w postaci (10)

The incompatibility of the estimate M $[S^*_{xx}(\omega)]$ and the true value $S_{xx}(\omega)$ results in the fact that the expressions for the variance of that deviation (or incompability) are tangled and unpractical. Under certain simplifying conditions the variance for the spectral density with the additional weighing function $\lambda(\tau)$ is

$$\sigma^2_{S^*}(\omega) = \frac{\tau_{max}}{T} S^2_{xx}(\omega)\, G(\lambda) \tag{16}$$

where

$$G(\lambda) = \int_{\infty -}^{+\infty} \lambda^2(\tau)\, d\tau \tag{17}$$

The weighing function $\lambda(\tau)$ optimizes the properties of the operation of integration, as in the general problem of the optimization of dynamic systems. If the weighing function is well selected, the estimate $S^*_{xx}(\omega)$ is as close to the true value $S_{xx}(\omega)$ as possible. It has been shown that the optimum weighing function should satisfy the condition

$$\lambda(\tau) = \frac{K^2_{xx}(\tau)}{K^2_{xx}(\tau) + \sigma^2_k(\tau)} \tag{18}$$

Upon consideration of the relative error (12) one can draw the following conclusions in respect of the choice of $\lambda(\tau)$: for long T, if τ_k is long, the values of Q_k are much smaller than 1 and then the weighing function is close to the optimum one. If the analysed series is short, for small α and long τ_{max} the function

$$\lambda(\tau) = \begin{cases} 1 - \left(\dfrac{\tau}{\tau_{max}}\right)^q & \text{for } 0 \leqslant \tau \leqslant \tau_{max} \\ 0 & \text{for } \tau > \tau_{max} \end{cases} \tag{19}$$

is almost optimum if q is equal to 1 for $\alpha = 0.23$. For high α and small τ_{max} the weighing function takes on the form (19) with $q > 2$. Examples

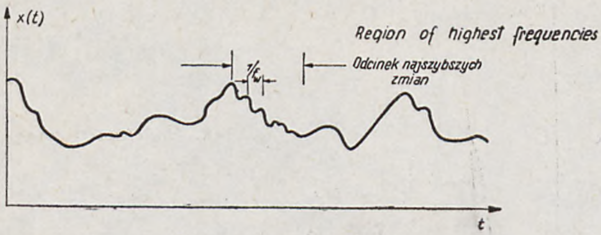

Fig. 7. Evaluation of the upper frequency ω_{max} from the graphical display of a process

Rys. 7. Sposób oceny górnej granicy argumentu widmowej ω_{max} na podstawie graficznego zapisu procesu

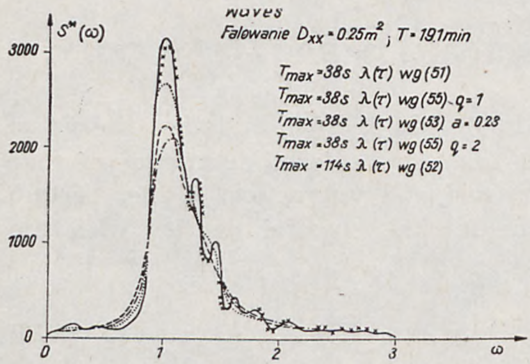

Fig. 8. Smoothing of the spectral density $S^*_{xx}(\omega)$ with different weighing functions

Rys. 8. Wygładzanie oceny funkcji gęstości widmowej $S^*_{xx}(\omega)$ za pomocą różnych funkcji wagowych

of various estimates of the spectral density are presented in Fig. 8. Hence, on the basis of the relationships presented above one can choose the optimum parameters T, τ_{max}, and Δ t and then select the weighing function which minimizes the deviation of the estimate of spectral density from the true value.

It is expedient to compare the above results with the accuracy of the Fast Fourier Transform, which is now widely used instead of the classical spectral analysis. The estimate of the spectral density in FFT is the square of the modulus of the Fourier coefficients $|A_k|^2$. The respective mean value is

$$M\,(|A_k|^2) = \int_{-\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} S\left(\frac{k}{L\,\Delta\,t} - f)\right) \frac{\sin^2 L\,\pi\cdot\Delta\,f\cdot f}{L\,\sin^2\pi\cdot\Delta\,f\cdot f}\,df \tag{20}$$

where

$$f = \frac{\omega}{2\pi}$$

$$S\,(f) = \Delta\,t \sum_{\tau=\infty}^{\tau=\infty} \cdot\,\varrho\,(\tau)\exp\,(-2\,\pi_i\cdot\tau\cdot\Delta\,t\cdot f)$$

$$\varrho\;(\tau) = M\,(X_t\cdot X_{t+\tau})$$

For the sequence X (j) taken from a normal process the variance of the estimate $\hat{S}(\omega_n)$ is

$$\varrho^2 \left\{\hat{S}(\omega_n)\right\} = \frac{\hat{S}(\omega_n)}{l} \left\{ 1 + 2 \sum_{j=1}^{L-1} \frac{l-j}{l} \varrho(j) \right\} \tag{21}$$

where

$$\varrho(j) = \left[\sum_{k=0}^{l-1} W(k)\, W(k+jD) \right]^2 : \left[\sum_{k=0}^{l-1} W^2(k) \right]^2$$

From the analysis of the above formula it follows that, if one has a sample with N=2048 and D=64, the standard deviation of the estimate $\hat{S}(\omega_n)$ is 22 percent smaller in cases of overlapping segments with $\frac{L}{2}$ than without overlapping segments.

Comparison of the the two methods of spectral analysis (classical and FFT), is shown in Fig. 3.

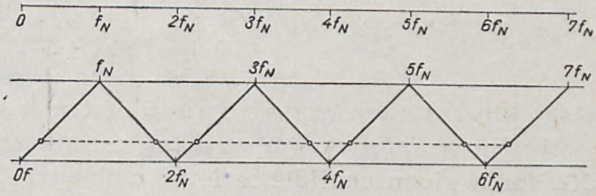

Fig. 9. "Parasitic" frequencies around the multiples of the Nyquist frequency

Rys. 9. Układanie się częstotliwości „pasożytniczych" wokół wielokrotności częstotliwości Nyquista

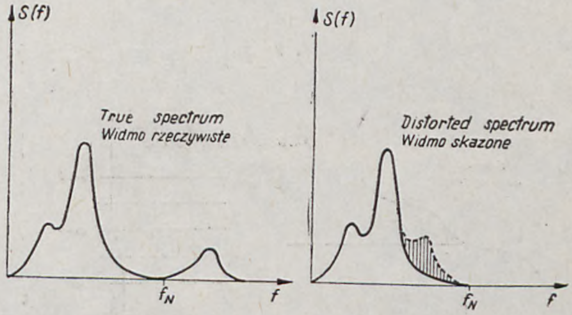

Fig. 10. Penetration of the "quantization noise" to the interval $f < f_N$

Rys. 10. Przenikanie „szumów dyskretyzacji" do przedziału $f < f_N$

Apart from the above mentioned, there are some other sources of errors in the estimations of spectral densities. If the highest frequency chosen is much below the Nyquist frequency, the energy borne in higher frequences is transferred artificially to the band of lower frequencies (Fig. 10). This so-called aliasing effect can be avoided by proper choice of the highest frequency in the spectral analysis or by filtering input data and rejecting the frequencies $f > f_N$.

Still another type of error, illustrated in Fig. 11, is connected with the inevitable choice of the quantization level. This error was evaluated for the manual analysis of the analogue records of the IBW-PAN wire--type wave gauges. The following correlation function of the error function was obtained:

$$K(\tau) = 0.1 \exp(-\alpha\,|\tau|)\ m^2;\ \alpha = 4.5\ s^{-1} \tag{22}$$

Since this function is close to the Dirac delta-function, the respective white noise arises in the estimate of spectral density.

For the reasons mentioned it is evident that the random characteristics discussed must lie within some confidence intervals. The statistical distributions of the mean value and variance of a random process are widely known. In the case of the Gaussian processes the distribution of $\sigma_{xx}^{*2} = K_{xx}^{*2}(0)$ is of χ^2 — type, while the standardized mean value $\frac{m_x^* - m_x}{\sigma_x}$ conforms to the Student distribution.

The analysis of the spectral density is more complex. It can be shown that the distribution of the ratio of spectral densities,

$$\frac{S^2}{S^{*2}} = \frac{\varepsilon^{*2}}{\varepsilon^2} : \frac{\sigma^{*2}}{\sigma_s^2} \tag{23}$$

is of the $F(n_1, n_2)$ type as a ratio of two variables with χ^2 distributions. Both degrees of freedom of $F(n_1, n_2)$ are the same. On the basis of Eq. 16., for a given confidence level α the true value S^2 must be contained in the interval

$$S^{*2} \cdot F_{n1, n2;\ 1-\frac{\alpha}{2}} < S^2 < S^{*2} F_{nt, o2;\ a/2} \tag{24}$$

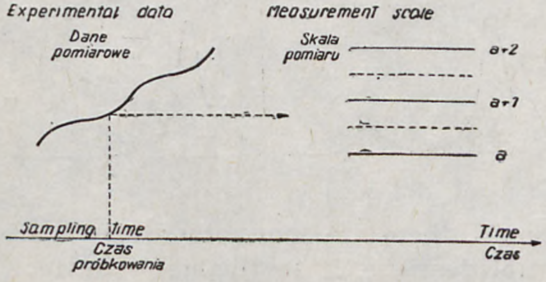

Fig. 11. The error of numerical resolution

Rys. 11. Błąd kwantowania

Other confidence intervals are presented in the Polish text (Studia i Mat. Ocean. 12, 1975).

In the light of the results presented herein the correlation-spectral analysis should proceed in the following steps:

i — choice of the quantization increment of the spectral density as per the requirement of the highest possible freedom

$$n = 2\,Be \cdot T = 2 \cdot \frac{1}{m\,\Delta t} N\,\Delta t = 2\,\frac{N}{m} \rightarrow \max. \qquad (25)$$

The maximum frequency f_N is to be found from an analogue record, as in Fig. 7, i.e. $f_N = 3 \div 4\, f_W$. Good representation of the spectral density is normally warranted by the time increment $\Delta t = \frac{2}{5\, f_N}$. At the same time, on the basis of Figs. 1 and 4 one has to choose Δt that gives a small error of the numerical approximation of mean value λ_m and variance λ_D.

ii — determination of the integration interval of the correlation function, $m\,\Delta t$. After assuming the quantization step of the spectral density, Be, (the spectral values separated by $\Delta f > Be$ are not correlated) m is given by

$$m = \frac{1}{Be\,\Delta t} \qquad (26)$$

The chosen value of m must be greater than $\frac{3}{\alpha\,\Delta t}$ and conform to the conditions of Eqs. 13 and 25.

iii — choice of the time length of the time series $N\,\Delta t$. The chosen value of N must allow for a high degree of freedom (after Eq. 10, 13, and 25) and must also be great enough to satisfy the stipulation $\alpha T \gg 1$. As a first approximation $N = \frac{m}{\varrho^2}$ can be taken, where ϱ^2 is the measure of the relative square error of spectral density.

Examples of the selection of the parameters of the above data processing procedure are given in the Polish text published in Studia i Materiały Oceanologiczne Nr 12, 1975.

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ESTYMATORY STOCHASTYCZNYCH PROCESÓW GEOFIZYCZNYCH

Streszczenie

Korelacyjne i widmowe charakterystyki procesu losowego powinny być obliczane w taki sposób, aby można było zminimalizować błędy wynikające z ich losowości, to jest rozbieżności w stosunku do prawdziwych charakterystyk populacji generalnej. Przeprowadzona w niniejszej pracy analiza tych rozbieżności i błędów obróbki danych pozwala wybrać parametry tych charakterystyk. Wybór ten musi być taki, aby krok kwantowania zapewniał maksymalizację stosunku długości próby czasowej T do elementarnego pasma widma $m\Delta t$. Maksymalna analizowana częstotliwość f_N powinna być trzy- czterokrotnie większa od najwyższej częstotliwości wykrytej na zapisie procesu. Wybrana wartość m powinna być większa od $\frac{3}{\alpha\Delta t}$ i ma spełniać wymagania zależności (3, 8), m powinno być dostatecznie duże, tak aby αT było znacznie większe od jedności.

Dokładność przekształcenia Fouriera przeanalizowano dla różnych postaci funkcji korelacji. Ważne jest, aby funkcja korelacji była przedłużana do przecięcia z osią odciętych.

Ważenie funkcji korelacji i gęstości widmowej powinno być optymalne. Zależność optymalnej funkcji wagi od postaci funkcji korelacji ilustruje równanie (18).

Porównanie wyników klasycznej analizy widmowej i szybkiej transformacji Fouriera FFT zilustrowano na rys. 3 oraz przedyskutowano w tekście zasadniczym.

W pracy określono także przedziały ufności podstawowych charakterystyk i omówiono specyficzne błędy.

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